Sensitivity analysis and optimal control for a contact problem with friction in the linear elastic model

Loïc Bourdin*, Fabien Caubet†, Aymeric Jacob de Cordemoy‡

June 5, 2023

Abstract

This paper investigates, without any regularization procedure, the sensitivity analysis of a mechanical contact problem involving the (nonsmooth) Tresca friction law in the linear elastic model. To this aim a recent methodology based on advanced tools from convex and variational analyses is used. Precisely we express the solution to the so-called Tresca friction problem thanks to the proximal operator associated with the corresponding Tresca friction functional. Then, using an extended version of twice epi-differentiability, we prove the differentiability of the solution to the parameterized Tresca friction problem, characterizing its derivative as the solution to a boundary value problem involving tangential Signorini's unilateral conditions. Finally our result is used to investigate and numerically solve an optimal control problem associated with the Tresca friction model.

Keywords: Sensitivity analysis, optimal control, contact mechanics, Tresca's friction law, Signorini's unilateral conditions, variational inequalities, proximal operator, twice epi-differentiability.

AMS Classification: 49J40, 74M10, 74M15, 35Q93.

1 Introduction

General context and motivation. On the one hand, *optimal control theory* is the mathematical field aimed at finding the control of a given system that allows to minimize a given cost while satisfying given constraints. In order to numerically solve an optimal control problem, the numerical descent methods usually require to compute the gradient of the cost functional which usually depends on the solution to a partial differential equation with given boundary conditions. Therefore a crucial point is to perform the *sensitivity analysis* of the solution to the boundary value problem with respect to perturbations, in order to characterize its derivative.

On the other hand, *contact mechanics* is the engineering field that describes the deformation of solids that touch each other on parts of their boundaries. A classical mechanical setting consists in a deformable body which is in contact with a rigid foundation, possibly sliding against it which causes friction on the contact surface. This friction can be mathematically modeled by the so-called *Tresca friction law* (see, e.g., [20]) which appears as a boundary condition involving nonsmooth inequalities depending on a friction threshold. Mechanical contact problems are usually investigated

*Institut de recherche XLIM. UMR CNRS 7252. Université de Limoges, France. loic.bourdin@unilim.fr

†Université de Pau et des Pays de l'Adour, E2S UPPA, CNRS, LMAP, UMR 5142, 64000 Pau, France. fabien.caubet@univ-pau.fr

‡Université de Pau et des Pays de l'Adour, E2S UPPA, CNRS, LMAP, UMR 5142, 64000 Pau, France. aymeric.jacob-de-cordemoy@univ-pau.fr

through the theory of *variational inequalities*, and the Tresca friction law causes nonlinearities and nonsmoothness in the corresponding variational formulations.

As a consequence, in order to investigate optimal control problems with mechanical contact models involving the Tresca friction law, we have to perform the sensitivity analysis of nonsmooth variational inequalities. The standard methods found in the literature usually consist in regularization (see, e.g., [7] or [19, Section 10.4 Chapter 10]) and dualization (see, e.g., [31, 32]) procedures. In a nutschell, regularization consists in replacing the nondifferentiable term by its Moreau's envelope to approximate the optimization problem associated with the model, thus the corresponding optimality condition is replaced by a smooth variational equality instead of a nonsmooth variational inequality. However this method does not take into account the exact characterization of the solution and perturbs the nonsmooth nature of the original physical model. The dualization method consists in describing the primal/dual pair of the model as a saddle point of the associated Lagrangian. The dual model leads to a characterization of its solution that involves only projection operators and thus Mignot's theorem (see [22]) about conical differentiability can be applied. With this method, the derivative of the solution to the primal model, with respect to perturbations, can be obtained but is characterized only implicitly, due to the presence of dual elements.

In this paper the sensitivity analysis is performed using a recent methodology, already used in our previous papers [3, 4, 10, 18], based on advanced tools from convex and variational analyses such as the notion of *proximal operator* introduced by J.J. Moreau in 1965 (see [24]) and the notion of *twice epi-differentiability* introduced by R.T. Rockafellar in 1985 (see [26]). This methodology allows us to preserve the original nonsmooth nature of the model, that is, without using any regularization procedure, and to work only with the primal model.

Objective and methodology. The present work follows from our previous papers [3, 10] in which the sensitivity analysis of boundary value problems involving the *scalar version* of the Tresca friction law are performed. In this new paper we focus on the classical Tresca friction law which is about the linear elastic model. Precisely we consider the (parameterized) *Tresca friction problem* given by

$$\left\{\begin{aligned} -\mathrm{div}(\mathrm{Ae}(u_t)) &= f_t & &\text{in } \Omega,\\ u_t &= 0 & &\text{on } \Gamma_{\mathrm{D}},\\ \sigma_{\mathrm{n}}(u_t) &= h_t & &\text{on } \Gamma_{\mathrm{N}},\\ \|\sigma_\tau(u_t)\| \leq g_t \text{ and } u_{t_\tau}\cdot\sigma_\tau(u_t) + g_t\,\|u_{t_\tau}\| &= 0 & &\text{on } \Gamma_{\mathrm{N}}, \end{aligned}\right. \tag{TP$_t$}$$

for all $t \geq 0$, where $\Omega \subset \mathbb{R}^d$ is a nonempty bounded connected open subset of $\mathbb{R}^d$, with $d \in \{2,3\}$ and with a $\mathcal{C}^1$-boundary denoted by $\Gamma := \partial\Omega$ (see Remark 3.17 for comments on this $\mathcal{C}^1$-regularity assumption), where n is the outward-pointing unit normal vector to Γ and where the boundary is decomposed as $\Gamma =: \Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{N}}$, where Γ_{D} and Γ_{N} are two measurable (with positive measure) pairwise disjoint subsets of Γ such that almost every point of Γ_{N} belongs to $\mathrm{int}_\Gamma(\Gamma_{\mathrm{N}})$ (see Remark 3.15 for comments on this last assumption). Recall that, in linear elasticity, A is the stiffness tensor, e is the infinitesimal strain tensor, σ_{n} is the normal stress and σ_τ is the shear stress (see Section 3 for details). Moreover $\|\cdot\|$ stands for the usual Euclidean norm of $\mathbb{R}^d$ and we assume that $f_t \in \mathrm{L}^2(\Omega,\mathbb{R}^d)$, $h_t \in \mathrm{L}^2(\Gamma_{\mathrm{N}})$ and $g_t \in \mathrm{L}^2(\Gamma_{\mathrm{N}})$, with $g_t > 0$ almost everywhere on Γ_{N}, for all $t \geq 0$. Finally we recall that the tangential boundary condition on Γ_{N} is known as the Tresca friction law. The main difference with respect to our previous paper [3] is that we work here on the linear elastic model, which implies several non-trivial technical adjustments, in particular for the computation of the twice epi-differentiability of the *Tresca friction functional* (see Subsubsection 3.2.1 for details).

The main objective of this work is to characterize the derivative of the map $t \in \mathbb{R}_+ \mapsto u_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ at $t = 0$, where $\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d) := \{w \in \mathrm{H}^1(\Omega,\mathbb{R}^d) \mid w = 0 \textit{ a.e.} \text{ on } \Gamma_{\mathrm{D}}\}$ and where the abbreviation *a.e.* stands for *almost everywhere*. However the norm $\|\cdot\|$ which appears in the Tresca friction law generates nonsmooth terms in the variational formulation of Problem (TP_t) given by: find $u_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ such that

$$\int_\Omega \mathrm{Ae}(u_t) : \mathrm{e}(w-u_t) + \int_{\Gamma_\mathrm{N}} g_t \|w_\tau\| - \int_{\Gamma_\mathrm{N}} g_t \|u_{t\tau}\| \geq \int_\Omega f_t \cdot (w-u_t) + \int_{\Gamma_\mathrm{N}} h_t (w_\mathrm{n} - u_{t\mathrm{n}}), \qquad \forall w \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d),$$

for all $t \geq 0$. Nevertheless recall that u_t can be expressed, using the proximal operator (see Definition 2.3), as

$$u_t = \mathrm{prox}_{\Phi(t,\cdot)}(F_t),$$

where $F_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ is the unique solution to the (smooth) *parameterized Dirichlet-Neumann problem* given by: find $F_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ such that

$$\int_\Omega \mathrm{Ae}(F_t) : \mathrm{e}(w) = \int_\Omega f_t \cdot w + \int_{\Gamma_\mathrm{N}} h_t w_\mathrm{n}, \qquad \forall w \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d),$$

for all $t \geq 0$, and where Φ is the *parameterized Tresca friction functional* defined by

$$\begin{array}{rrcl} \Phi: & \mathbb{R}_+ \times \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d) & \longrightarrow & \mathbb{R} \\ & (t,w) & \longmapsto & \Phi(t,w) := \displaystyle\int_{\Gamma_\mathrm{N}} g_t \|w_\tau\|. \end{array}$$

Similarly to our previous paper [10], to deal with the differentiability (in a generalized sense) of the parameterized proximal operator $\mathrm{prox}_{\Phi(t,\cdot)} : \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d) \to \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$, we will invoke the notion of twice epi-differentiability for convex functions introduced by R.T. Rockafellar in 1985 (see [26]) which leads to the *protodifferentiability* of the corresponding proximal operators. Actually, since the work by R.T. Rockafellar deals only with nonparameterized convex functions, we will use instead the recent work [2] in which the notion of twice epi-differentiability has been extended to parameterized convex functions (see Definition 2.12).

Main result. With the previous methodology and under some appropriate assumptions described in Theorem 3.23, we prove that the map $t \in \mathbb{R}_+ \mapsto u_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ is differentiable at $t = 0$, and its derivative $u'_0 \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ is given by

$$u'_0 = \mathrm{prox}_{\mathrm{D}^2_e\Phi(u_0|F_0-u_0)}(F'_0),$$

where $\mathrm{D}^2_e\Phi(u_0|F_0-u_0)$ stands for the second-order epi-derivative (see Definition 2.12) of the parameterized Tresca friction functional Φ at u_0 for $F_0 - u_0$, and where $F'_0 \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ is the derivative at $t = 0$ of the map $t \in \mathbb{R}_+ \mapsto F_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$. Moreover we prove that $u'_0 \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ exactly corresponds to the unique weak solution to the *tangential Signorini problem*

$$\left\{ \begin{array}{rcll} -\mathrm{div}(\mathrm{Ae}(u'_0)) & = & f'_0 & \text{in } \Omega, \\ u'_0 & = & 0 & \text{on } \Gamma_\mathrm{D}, \\ \sigma_\mathrm{n}(u'_0) & = & h'_0 & \text{on } \Gamma_\mathrm{N}, \\ u'_{0_\tau} & = & 0 & \text{on } \Gamma_{\mathrm{N}^{u_0,g_0}_\mathrm{T}}, \\ \sigma_\tau(u'_0) + \frac{g_0}{\|u_{0\tau}\|}\left(u'_{0_\tau} - \left(u'_{0_\tau} \cdot \frac{u_{0\tau}}{\|u_{0\tau}\|}\right)\frac{u_{0\tau}}{\|u_{0\tau}\|}\right) & = & -g'_0 \frac{u_{0\tau}}{\|u_{0\tau}\|} & \text{on } \Gamma_{\mathrm{N}^{u_0,g_0}_\mathrm{R}}, \\ u'_{0_\tau} \in \mathbb{R}_- \frac{\sigma_\tau(u_0)}{g_0}, \left(\sigma_\tau(u'_0) - g'_0 \frac{\sigma_\tau(u_0)}{g_0}\right) \cdot \frac{\sigma_\tau(u_0)}{g_0} \leq 0 & & & \\ \text{and } u'_{0_\tau} \cdot \left(\sigma_\tau(u'_0) - g'_0 \frac{\sigma_\tau(u_0)}{g_0}\right) & = & 0 & \text{on } \Gamma_{\mathrm{N}^{u_0,g_0}_\mathrm{S}}, \end{array} \right.$$

where Γ_{N} is decomposed (up to a null set) as $\Gamma_{\mathrm{N}_{\mathrm{T}}^{u_0,g_0}} \cup \Gamma_{\mathrm{N}_{\mathrm{R}}^{u_0,g_0}} \cup \Gamma_{\mathrm{N}_{\mathrm{S}}^{u_0,g_0}}$ (see details in Theorem 3.23), and where, for almost all $s \in \Gamma_{\mathrm{N}_{\mathrm{S}}^{u_0,g_0}}$, $\mathbb{R}_{-}\frac{\sigma_\tau(u_0)(s)}{g_0(s)} := \{y \in \mathbb{R}^d \mid \exists \nu \leq 0 \text{ such that } y = \nu \frac{\sigma_\tau(u_0)(s)}{g_0(s)}\}$. Here $f_0' \in \mathrm{L}^2(\Omega,\mathbb{R}^d)$ (resp. $h_0' \in \mathrm{L}^2(\Gamma_{\mathrm{N}})$) is the derivative at $t=0$ of the map $t \in \mathbb{R}_+ \mapsto f_t \in \mathrm{L}^2(\Omega,\mathbb{R}^d)$ (resp. of the map $t \in \mathbb{R}_+ \mapsto h_t \in \mathrm{L}^2(\Gamma_{\mathrm{N}})$) and $g_0' \in \mathrm{L}^2(\Gamma_{\mathrm{N}})$ is the map defined, for almost every $s \in \Gamma_{\mathrm{N}}$, by $g_0'(s) := \lim_{t\to 0^+} \frac{g_t(s)-g_0(s)}{t}$.

We emphasize that the boundary conditions which appear on $\Gamma_{\mathrm{N}_{\mathrm{S}}^{u_0,g_0}}$ are called the *tangential Signorini's unilateral conditions*. They are close to the classical Signorini's unilateral conditions which describe a non-permeable contact (see, e.g., [29, 30]) except that, here, they are concerned with the tangential components (instead of the usual normal components). Roughly speaking our main result claims that the tangential Signorini's solution can be considered as first-order approximation to the perturbed Tresca's solution.

Application to an optimal control problem. The above sensitivity analysis allows us to investigate the optimal control problem given by

$$\underset{z\in\mathcal{U}}{\text{minimize}}\ \mathcal{J}(z),$$

where $\mathcal{J}$ is the cost functional given by

$$\begin{array}{rccl} \mathcal{J}: & \mathrm{V} & \longrightarrow & \mathbb{R} \\ & z & \longmapsto & \mathcal{J}(z) := \frac{1}{2}\left\|u(\ell(z))\right\|^2_{\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)} + \frac{\beta}{2}\left\|\ell(z)\right\|^2_{\mathrm{L}^2(\Gamma_{\mathrm{N}})}, \end{array}$$

where V is the open subset of $\mathrm{L}^\infty(\Gamma_{\mathrm{N}})$ defined by

$$\mathrm{V} := \left\{z \in \mathrm{L}^\infty(\Gamma_{\mathrm{N}}) \mid \exists C(z) > 0,\, \ell(z) > C(z) \text{ } a.e. \text{ on } \Gamma_{\mathrm{N}}\right\},$$

where ℓ is the map defined by $z \in \mathrm{L}^\infty(\Gamma_{\mathrm{N}}) \mapsto \ell(z) := g_1 + zg_2 \in \mathrm{L}^\infty(\Gamma_{\mathrm{N}})$, where $g_1 \in \mathrm{L}^\infty(\Gamma_{\mathrm{N}})$ with $g_1 \geq m$ *a.e.* on Γ_{N} for some positive constant $m > 0$ and $g_2 \in \mathrm{L}^\infty(\Gamma_{\mathrm{N}})$ such that $||g_2||_{\mathrm{L}^\infty(\Gamma_{\mathrm{N}})} > 0$, and where $u(\ell(z)) \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ stands for the unique solution to the Tresca friction problem given by

$$\left\{\begin{aligned} -\mathrm{div}(\mathrm{Ae}(u)) &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma_{\mathrm{D}}, \\ \sigma_{\mathrm{n}}(u) &= h && \text{on } \Gamma_{\mathrm{N}}, \\ \|\sigma_\tau(u)\| \leq \ell(z) \text{ and } u_\tau \cdot \sigma_\tau(u) + \ell(z)\left\|u_\tau\right\| &= 0 && \text{on } \Gamma_{\mathrm{N}}, \end{aligned}\right. \tag{CTP$_{\ell(z)}$}$$

where $f \in \mathrm{L}^2(\Omega,\mathbb{R}^d)$ and $h \in \mathrm{L}^2(\Gamma_{\mathrm{N}})$, where $\beta > 0$ is a positive constant and where $\mathcal{U}$ is a given nonempty convex subset of V such that $\mathcal{U}$ is a bounded closed subset of $\mathrm{L}^2(\Gamma_{\mathrm{N}})$. Note that the first term in the cost functional $\mathcal{J}$ corresponds to the compliance, while the second term is the energy consumption which is standard in optimal control problems (see, e.g., [21]).

We prove in Theorem 4.3 that the cost functional $\mathcal{J}$ is Gateaux differentiable on V, and its Gateaux differential at any $z_0 \in \mathrm{V}$, denoted by $\mathrm{d}_G\mathcal{J}(z_0)$, is given by

$$\mathrm{d}_G\mathcal{J}(z_0)(z) = \int_{\Gamma_{\mathrm{N}_{\mathrm{R}}^{u_0,\ell(z_0)}}} zg_2\left(\beta\left(g_1+z_0g_2\right) - \|u_{0_\tau}\|\right) + \int_{\Gamma_{\mathrm{N}_{\mathrm{T}}^{u_0,\ell(z_0)}} \cup \Gamma_{\mathrm{N}_{\mathrm{S}}^{u_0,\ell(z_0)}}} \beta zg_2\left(g_1+z_0g_2\right),$$

for all $z \in \mathrm{L}^\infty(\Gamma_{\mathrm{N}})$, where $u_0 := u(\ell(z_0))$ is the solution to the Tresca friction problem (CTP$_{\ell(z_0)}$), and where Γ_{N} is decomposed (up to a null set) as $\Gamma_{\mathrm{N}_{\mathrm{T}}^{u_0,\ell(z_0)}} \cup \Gamma_{\mathrm{N}_{\mathrm{R}}^{u_0,\ell(z_0)}} \cup \Gamma_{\mathrm{N}_{\mathrm{S}}^{u_0,\ell(z_0)}}$.

The expression of the Gateaux differential of $\mathcal{J}$ allows us to exhibit an explicit descent direction of $\mathcal{J}$ (see Subsection 4.3 for details). Hence, using this descent direction together with a basic projected gradient algorithm, we perform numerical simulations to solve the optimal control problem on a two-dimensional example.

Organization of the paper. The paper is organized as follows. Section 2 is dedicated to some basic recalls from convex, variational and functional analyses used throughout the paper. Section 3 is the core of the present work: in Subsection 3.1 we introduce three general boundary value problems that are involved all along the paper; in Subsection 3.2 the sensitivity analysis of the Tresca friction problem is performed. Finally, in Section 4, we investigate an optimal control problem and numerical simulations are performed to solve it on a two-dimensional example.

2 Preliminaries

2.1 Reminders from convex and variational analyses

For notions and results presented in this section, we refer to standard references such as [11, 23, 25] and [27, Chapter 12]. In what follows $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ stands for a general real Hilbert space.

Definition 2.1 (Domain and epigraph)**.** *Let $\phi : \mathcal{H} \to \mathbb{R} \cup \{\pm\infty\}$. The* domain *and the* epigraph *of ϕ are respectively defined by*

$$\mathrm{dom}\,(\phi) := \{x \in \mathcal{H} \mid \phi(x) < +\infty\} \quad \textit{and} \quad \mathrm{epi}\,(\phi) := \{(x,\nu) \in \mathcal{H} \times \mathbb{R} \mid \phi(x) \leq \nu\}\,.$$

Recall that $\phi : \mathcal{H} \to \mathbb{R}\cup\{\pm\infty\}$ is said to be *proper* if $\mathrm{dom}(\phi) \neq \emptyset$ and $\phi(x) > -\infty$ for all $x \in \mathcal{H}$. Moreover, ϕ is a convex (resp. lower semi-continuous) function on $\mathcal{H}$ if and only if $\mathrm{epi}(\phi)$ is a convex (resp. closed) subset of $\mathcal{H} \times \mathbb{R}$.

Definition 2.2 (Convex subdifferential operator)**.** *Let $\phi : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ be a proper function. We denote by $\partial\phi : \mathcal{H} \rightrightarrows \mathcal{H}$ the* convex subdifferential operator *of ϕ, defined by*

$$\partial\phi(x) := \{y \in \mathcal{H} \mid \forall z \in \mathcal{H},\ \langle y, z-x\rangle_{\mathcal{H}} \leq \phi(z) - \phi(x)\}\,,$$

for all $x \in \mathcal{H}$.

Definition 2.3 (Proximal operator)**.** *Let $\phi : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semi-continuous convex function. The* proximal operator *associated with ϕ is the map $\mathrm{prox}_{\phi} : \mathcal{H} \to \mathcal{H}$ defined by*

$$\mathrm{prox}_{\phi}(x) := \underset{y\in\mathcal{H}}{\mathrm{argmin}} \left[\phi(y) + \frac{1}{2}\left\|y-x\right\|^2_{\mathcal{H}}\right] = (\mathrm{I} + \partial\phi)^{-1}(x),$$

for all $x \in \mathcal{H}$, where $\mathrm{I} : \mathcal{H} \to \mathcal{H}$ stands for the identity operator.

The proximal operator have been introduced by J.-J. Moreau in 1965 (see [24]) and can be seen as a generalization of the classical projection operators onto nonempty closed convex subsets. It is well-known that, if $\phi : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ is a proper lower semi-continuous convex function, then $\partial\phi$ is a maximal monotone operator (see, e.g., [25]), and thus the proximal operator prox_{ϕ} is well-defined and a single-valued map (see, e.g., [11, Chapter II]).

We pursue with the following classical result which will be crucial to prove the existence of a unique weak solution to the tangential Signorini problem (see Proposition 3.10).

Proposition 2.4. *Let $\phi : \mathcal{H} \to \mathbb{R}$ be a Fréchet differentiable convex function and C be a nonempty convex subset of $\mathcal{H}$. Let $y \in \mathrm{C}$ and $x \in \mathcal{H}$. Then the following variational inequalities are equivalent:*

(i) $\varphi(z) - \varphi(y) \geq \langle x-y, z-y\rangle_{\mathcal{H}}\,, \qquad \forall z \in \mathrm{C};$

(ii) $\langle \nabla\varphi(y), z-y\rangle_{\mathcal{H}} \geq \langle x-y, z-y\rangle_{\mathcal{H}}\,, \qquad \forall z \in \mathrm{C}.$

In what follows, some definitions related to the notion of twice epi-differentiability are recalled (for more details, see [27, Chapter 7, section B p.240] for the finite-dimensional case and [14] for the infinite-dimensional one). The strong (resp. weak) convergence of a sequence in $\mathcal{H}$ will be denoted by $\to$ (resp. $\rightharpoonup$) and note that all limits with respect to t will be considered for $t \to 0^+$.

Definition 2.5 (Mosco-convergence)**.** *The* outer*,* weak-outer*,* inner *and* weak-inner limits *of a parameterized family $(A_t)_{t>0}$ of subsets of $\mathcal{H}$ are respectively defined by*

$$\begin{aligned}
\limsup A_t &:= \left\{ x \in \mathcal{H} \mid \exists (t_n)_{n\in\mathbb{N}} \to 0^+, \exists\, (x_n)_{n\in\mathbb{N}} \to x, \forall n \in \mathbb{N}, x_n \in A_{t_n} \right\}, \\
\text{w-}\limsup A_t &:= \left\{ x \in \mathcal{H} \mid \exists (t_n)_{n\in\mathbb{N}} \to 0^+, \exists\, (x_n)_{n\in\mathbb{N}} \rightharpoonup x, \forall n \in \mathbb{N}, x_n \in A_{t_n} \right\}, \\
\liminf A_t &:= \left\{ x \in \mathcal{H} \mid \forall (t_n)_{n\in\mathbb{N}} \to 0^+, \exists\, (x_n)_{n\in\mathbb{N}} \to x, \exists N \in \mathbb{N}, \forall n \geq N, x_n \in A_{t_n} \right\}, \\
\text{w-}\liminf A_t &:= \left\{ x \in \mathcal{H} \mid \forall (t_n)_{n\in\mathbb{N}} \to 0^+, \exists\, (x_n)_{n\in\mathbb{N}} \rightharpoonup x, \exists N \in \mathbb{N}, \forall n \geq N, x_n \in A_{t_n} \right\}.
\end{aligned}$$

The family $(A_t)_{t>0}$ is said to be Mosco-convergent *if* $\text{w-}\limsup A_t \subset \liminf A_t$*. In that case all the previous limits are equal and we write*

$$\text{M-lim}\, A_t := \liminf A_t = \limsup A_t = \text{w-}\liminf A_t = \text{w-}\limsup A_t.$$

Definition 2.6 (Mosco epi-convergence)**.** *Let $(\phi_t)_{t>0}$ be a parameterized family of functions $\phi_t : \mathcal{H} \to \mathbb{R} \cup \{\pm\infty\}$ for all $t > 0$. We say that $(\phi_t)_{t>0}$ is* Mosco epi-convergent *if* $(\text{epi}(\phi_t))_{t>0}$ *is Mosco-convergent in $\mathcal{H} \times \mathbb{R}$. Then we denote by* $\text{ME-lim}\ \phi_t : \mathcal{H} \to \mathbb{R} \cup \{\pm\infty\}$ *the function characterized by its epigraph* $\text{epi}\,(\text{ME-lim}\ \phi_t) := \text{M-lim epi}\,(\phi_t)$ *and we say that $(\phi_t)_{t>0}$ Mosco epi-converges to* $\text{ME-lim}\ \phi_t$*.*

The proof of the next proposition can be found in [8, Proposition 3.19 p.297].

Proposition 2.7 (Characterization of Mosco epi-convergence)**.** *Let $(\phi_t)_{t>0}$ be a parameterized family of functions $\phi_t : \mathcal{H} \to \mathbb{R} \cup \{\pm\infty\}$ for all $t > 0$ and let $\phi : \mathcal{H} \to \mathbb{R} \cup \{\pm\infty\}$. Then $(\phi_t)_{t>0}$ Mosco epi-converges to ϕ if and only if, for all $x \in \mathcal{H}$, the two conditions:*

(i) *there exists $(x_t)_{t>0} \to x$ such that* $\limsup \phi_t(x_t) \leq \phi(x)$*;*

(ii) *for all $(x_t)_{t>0} \rightharpoonup x$,* $\liminf \phi_t(x_t) \geq \phi(x)$*;*

are both satisfied.

Now let us recall the notion of twice epi-differentiability introduced by R.T. Rockafellar in 1985 (see [26]) that generalizes the classical notion of second-order derivative to nonsmooth convex functions.

Definition 2.8 (Twice epi-differentiability)**.** *A proper lower semi-continuous convex function $\phi : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ is said to be* twice epi-differentiable *at $x \in \text{dom}(\phi)$ for $y \in \partial\phi(x)$ if the family of second-order difference quotient functions $(\delta_t^2\phi(x \mid y))_{t>0}$ defined by*

$$\begin{array}{rccl}
\delta_t^2\phi(x \mid y) : & \mathcal{H} & \longrightarrow & \mathbb{R} \cup \{+\infty\} \\
& z & \longmapsto & \dfrac{\phi(x + tz) - \phi(x) - t\,\langle y, z\rangle_{\mathcal{H}}}{t^2},
\end{array}$$

for all $t > 0$, is Mosco epi-convergent. In that case we denote by

$$\mathrm{d}_e^2\phi(x \mid y) := \text{ME-lim}\ \delta_t^2\phi(x \mid y),$$

which is called the second-order epi-derivative *of ϕ at x for y.*

Remark 2.9. In the case where ϕ is twice Fréchet differentiable at $x \in \mathcal{H}$, then ϕ is twice epi-differentiable at x for $\nabla\phi(x)$ and

$$\mathrm{d}_e^2\phi(x \mid \nabla\phi(x))(z) = \frac{1}{2}\mathrm{D}^2\phi(x)(z,z), \qquad \forall z \in \mathcal{H},$$

where $\mathrm{D}^2\phi(x)$ stands for the second-order Fréchet differential of ϕ at x. Note that the factor $\frac{1}{2}$ could be removed if the family of second-order difference quotient functions is defined with a factor $\frac{1}{2}$ in the denominator (see the original definition in [26]).

Before proving the twice epi-differentiability of the support function of a nonempty closed convex set, let us recall the definition of the normal cone.

Definition 2.10 (Normal cone)**.** *Let* C *be a nonempty closed convex subset of* $\mathcal{H}$*. The normal cone to* C *at* $y \in \mathrm{C}$ *is the nonempty closed convex cone of* $\mathcal{H}$ *defined by*

$$\mathrm{N_C}(y) := \{z \in \mathcal{H} \mid \langle z, c-y\rangle_{\mathcal{H}} \leq 0, \forall c \in \mathrm{C}\}.$$

The following proposition is an extension of a result proved in [14, Example 2.7 p.286] that will be useful in Section 3 to compute the twice epi-differentiability of the tangential norm map (see Lemma 3.20).

Proposition 2.11. *Let* ξ_C *be the support function of a nonempty closed convex subset* C *of* $\mathcal{H}$ *defined by*

$$\begin{array}{rccl} \xi_\mathrm{C}: & \mathcal{H} & \longrightarrow & \mathbb{R}\cup\{+\infty\} \\ & x & \longmapsto & \xi_\mathrm{C}(x) := \sup\limits_{y\in\mathrm{C}} \langle x, y\rangle_{\mathcal{H}}. \end{array}$$

Then, for all $x \in \mathrm{C}^\perp := \{z \in \mathcal{H} \mid \langle z, c\rangle_{\mathcal{H}} = 0, \forall c \in \mathrm{C}\}$*, one has* $\partial\xi_\mathrm{C}(x) = \mathrm{C}$ *and* ξ_C *is twice epi-differentiable at* x *for any* $y \in \mathrm{C}$ *with*

$$\mathrm{d}_e^2\xi_\mathrm{C}(x \mid y) = \iota_{\mathrm{N_C}(y)},$$

where $\iota_{\mathrm{N_C}(y)}$ *stands for the indicator function of* $\mathrm{N_C}(y)$ *defined by* $\iota_{\mathrm{N_C}(y)}(z) := 0$ *if* $z \in \mathrm{N_C}(y)$*, and* $\iota_{\mathrm{N_C}(y)}(z) := +\infty$ *otherwise.*

Proof. Let $x \in \mathrm{C}^\perp$. From [17, Lesson E], it holds that:

(i) $y \in \partial\xi_\mathrm{C}(x) \Leftrightarrow x \in \partial\iota_\mathrm{C}(y) \Leftrightarrow y \in \mathrm{C}$;

(ii) if $y \in \mathrm{C}$, then $z \in \mathrm{N_C}(y) \Leftrightarrow \xi_\mathrm{C}(z) = \langle z, y\rangle_{\mathcal{H}}$.

From the first item one deduces that $\partial\xi_\mathrm{C}(x) = \mathrm{C}$. Let $y \in \mathrm{C}$ and let us prove that h_C is twice epi-differentiable at x for y. To this aim we use Proposition 2.7. Consider $z \in \mathrm{N_C}(y)$ and thus $\xi_\mathrm{C}(z) = \langle y, z\rangle$. By considering the sequence $z_t := z$ for all $t > 0$, one gets

$$\delta_t^2\xi_\mathrm{C}(x \mid y)(z_t) = \frac{\xi_\mathrm{C}(x+tz) - \xi_\mathrm{C}(x) - t\langle y, z\rangle_{\mathcal{H}}}{t^2} = \\ \frac{\sup_{c\in\mathrm{C}}\langle x+tz, c\rangle_{\mathcal{H}} - t\langle y, z\rangle_{\mathcal{H}}}{t^2} = \frac{\xi_\mathrm{C}(z) - \xi_\mathrm{C}(z)}{t} = 0.$$

Moreover, since $\delta_t^2\xi_\mathrm{C}(x \mid y)(v) \geq 0$ for all $v \in \mathcal{H}$, one deduces that $\mathrm{d}_e^2\xi_\mathrm{C}(x \mid y)(z) = 0$. Now consider $z \notin \mathrm{N_C}(y)$. There exists $c_0 \in \mathrm{C}$ such that $\langle z, c_0\rangle_{\mathcal{H}} > \langle z, y\rangle_{\mathcal{H}}$, thus $\xi_\mathrm{C}(z) > \langle z, y\rangle_{\mathcal{H}}$.

Consider any sequence $(z_t)_{t>0} \rightharpoonup z$. Since ξ_{C} is convex and lower semi-continuous, then ξ_{C} is also weakly lower semi-continuous (see, e.g., [12, Corollary 3.9 p.61]), thus one has

$$\liminf \xi_{\mathrm{C}}(z_t) \geq \xi_{\mathrm{C}}(z) \quad \text{and} \quad \langle y, z_t \rangle_{\mathcal{H}} \longrightarrow \langle y, z \rangle_{\mathcal{H}},$$

when $t \to 0^+$. Therefore there exists $\varepsilon > 0$ such that

$$\xi_{\mathrm{C}}(z_t) \geq \xi_{\mathrm{C}}(z) - \frac{\xi_{\mathrm{C}}(z) - \langle y, z \rangle_{\mathcal{H}}}{4} \quad \text{and} \quad -\langle y, z_t \rangle_{\mathcal{H}} \geq -\langle y, z \rangle_{\mathcal{H}} - \frac{\xi_{\mathrm{C}}(z) - \langle y, z \rangle_{\mathcal{H}}}{4},$$

for all $t \leq \varepsilon$, and thus

$$\delta_t^2 \xi_{\mathrm{C}}(x \mid y)(z_t) = \frac{\xi_{\mathrm{C}}(z_t) - \langle y, z_t \rangle_{\mathcal{H}}}{t} \geq \frac{\xi_{\mathrm{C}}(z) - \langle y, z \rangle_{\mathcal{H}}}{2t} \longrightarrow +\infty,$$

when $t \to 0^+$. Hence $\mathrm{d}_e^2 \xi_{\mathrm{C}}(x \mid y)(z) = +\infty$ which concludes the proof. □

In the above classical definition of twice epi-differentiability, the function ϕ does not depend on the parameter t. However, in this paper, the parameterized Tresca friction functional does (see Introduction). Therefore we will use an extended version of twice epi-differentiability which has been recently introduced in [2]. To this aim, when considering a function $\Phi : \mathbb{R}_+ \times \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ such that, for all $t \geq 0$, $\Phi(t, \cdot)$ is a proper function on $\mathcal{H}$, we will make use of the two following notations: $\partial\Phi(0, \cdot)(x)$ stands for the convex subdifferential operator at $x \in \mathcal{H}$ of the map $w \in \mathcal{H} \mapsto \Phi(0, w) \in \mathbb{R} \cup \{+\infty\}$, and $\Phi^{-1}(\cdot, \mathbb{R}) := \{x \in \mathcal{H} \mid \forall t \geq 0,\ \Phi(t, x) \in \mathbb{R}\}$.

Definition 2.12 (Twice epi-differentiability depending on a parameter)**.** *Let $\Phi : \mathbb{R}_+ \times \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ be a function such that, for all $t \geq 0$, $\Phi(t, \cdot)$ is a proper lower semi-continuous convex function on $\mathcal{H}$. The function Φ is said to be* twice epi-differentiable *at $x \in \Phi^{-1}(\cdot, \mathbb{R})$ for $y \in \partial\Phi(0, \cdot)(x)$ if the family of second-order difference quotient functions $(\Delta_t^2 \Phi(x \mid y))_{t>0}$ defined by*

$$\begin{array}{rccl} \Delta_t^2 \Phi(x \mid y) : & \mathcal{H} & \longrightarrow & \mathbb{R} \cup \{+\infty\} \\ & z & \longmapsto & \dfrac{\Phi(t, x + tz) - \Phi(t, x) - t \langle y, z \rangle_{\mathcal{H}}}{t^2}, \end{array}$$

for all $t > 0$, is Mosco epi-convergent. In that case, we denote by

$$\mathrm{D}_e^2 \Phi(x \mid y) := \text{ME-lim } \Delta_t^2 \Phi(x \mid y),$$

which is called the second-order epi-derivative *of Φ at x for y.*

Note that, if the function Φ is t-independent in Definition 2.12, then we recover Definition 2.8. Finally the following theorem is the key point in order to derive our main result in this paper. It is a particular case of a more general theorem that can be found in [2, Theorem 4.15 p.1714].

Theorem 2.13. *Let $\Phi : \mathbb{R}_+ \times \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ be a function such that, for all $t \geq 0$, $\Phi(t, \cdot)$ is a proper lower semi-continuous convex function on $\mathcal{H}$. Let $F : \mathbb{R}_+ \to \mathcal{H}$ and let $u : \mathbb{R}_+ \to \mathcal{H}$ be defined by*

$$u(t) := \mathrm{prox}_{\Phi(t,\cdot)}(F(t)),$$

for all $t \geq 0$. If the conditions:

(i) *F is differentiable at $t = 0$;*

(ii) *Φ is twice epi-differentiable at $u(0)$ for $F(0) - u(0) \in \partial\Phi(0, \cdot)(u(0))$;*

(iii) *$\mathrm{D}_e^2 \Phi(u(0) | F(0) - u(0))$ is a proper function on $\mathcal{H}$;*

are satisfied, then u is differentiable at $t = 0$ with

$$u'(0) = \mathrm{prox}_{\mathrm{D}_e^2 \Phi(u(0) | F(0) - u(0))}(F'(0)).$$

2.2 Functional framework

Let $d \in \{2,3\}$ and Ω be a nonempty bounded connected open subset of $\mathbb{R}^d$ with a $\mathcal{C}^1$-boundary $\Gamma := \partial\Omega$ and n be the outward-pointing unit normal vector to Γ. We denote by $\mathrm{L}^2(\Omega,\mathbb{R}^d)$, $\mathrm{L}^2(\Gamma,\mathbb{R}^d)$, $\mathrm{L}^1(\Gamma,\mathbb{R}^d)$, $\mathrm{H}^1(\Omega,\mathbb{R}^d)$, $\mathrm{H}^{1/2}(\Gamma,\mathbb{R}^d)$, $\mathrm{H}^{-1/2}(\Gamma,\mathbb{R}^d)$ the usual Lebesgue and Sobolev spaces endowed with their standard norms. Moreover the notation $\mathcal{D}(\Omega,\mathbb{R}^d)$ stands for the set of infinitely differentiable functions $\varphi : \Omega \to \mathbb{R}^d$ with compact support in Ω, and $\mathcal{D}'(\Omega,\mathbb{R}^d)$ for the set of distributions on Ω. Moreover, all along this paper, we denote by : the scalar product defined by $\mathrm{B} : \mathrm{C} = \sum_{i=1}^{d} \mathrm{B}_i \cdot \mathrm{C}_i$ for all $\mathrm{B},\mathrm{C} \in \mathbb{R}^{d\times d}$, where $\mathrm{B}_i \in \mathbb{R}^d$ (resp. $\mathrm{C}_i \in \mathbb{R}^d$) is the i-th line of B (resp. C) for all $i \in [[1,d]]$. In what follows we consider a decomposition $\Gamma =: \Gamma_1 \cup \Gamma_2$ where Γ_1 and Γ_2 are two measurable disjoint subsets of Γ. Let us recall some embeddings useful in this work, that can be found for instance in [1, Chapter 4, p.79], [9], [12], and [13, Chapter 7, Section 2 p.395].

Proposition 2.14. *The continuous and dense embeddings:*

- $\mathrm{H}^1(\Omega,\mathbb{R}^d) \hookrightarrow \mathrm{H}^{1/2}(\Gamma,\mathbb{R}^d) \hookrightarrow \mathrm{L}^2(\Gamma,\mathbb{R}^d) \hookrightarrow \mathrm{H}^{-1/2}(\Gamma,\mathbb{R}^d)$;
- $\mathrm{L}^2(\Gamma,\mathbb{R}^d) \hookrightarrow \mathrm{L}^1(\Gamma,\mathbb{R}^d)$;
- $\mathrm{H}^1(\Omega,\mathbb{R}^d) \hookrightarrow \mathrm{L}^2(\Omega,\mathbb{R}^d)$;
- $\mathrm{H}^{1/2}_{00}(\Gamma_1,\mathbb{R}^d) \hookrightarrow \mathrm{L}^2(\Gamma_1,\mathbb{R}^d) \hookrightarrow \mathrm{H}^{-1/2}_{00}(\Gamma_1,\mathbb{R}^d)$;

are satisfied, where $\mathrm{H}^{1/2}_{00}(\Gamma_1,\mathbb{R}^d)$ can be identified to a linear subspace of $\mathrm{H}^{1/2}(\Gamma,\mathbb{R}^d)$ defined by

$$\mathrm{H}^{1/2}_{00}(\Gamma_1,\mathbb{R}^d) := \left\{ w \in \mathrm{L}^2(\Gamma_1,\mathbb{R}^d) \mid \exists v \in \mathrm{H}^1(\Omega,\mathbb{R}^d),\ v = w \text{ a.e. on } \Gamma_1 \text{ and } v = 0 \text{ a.e. on } \Gamma_2 \right\},$$

and $\mathrm{H}^{-1/2}_{00}(\Gamma_1,\mathbb{R}^d)$ stands for its dual space. Furthermore the dense and compact embedding

$$\mathrm{H}^1(\Omega,\mathbb{R}^d) \hookrightarrow\!\!\!\!\rightarrow \mathrm{L}^2(\Gamma,\mathbb{R}^d),$$

holds true, and since $d \in \{2,3\}$, then we have the continuous embedding $\mathrm{H}^1(\Omega,\mathbb{R}^d) \hookrightarrow \mathrm{L}^4(\Gamma,\mathbb{R}^d)$.

The next proposition is a particular case of a more general statement that can be found in [33, Section 2.9 p.56].

Proposition 2.15. *Let $v \in \mathrm{H}^{-1/2}_{00}(\Gamma_1,\mathbb{R}^d)$. If there exists $C \geq 0$ such that*

$$\langle v, w \rangle_{\mathrm{H}^{-1/2}_{00}(\Gamma_1,\mathbb{R}^d) \times \mathrm{H}^{1/2}_{00}(\Gamma_1,\mathbb{R}^d)} \leq C \|w\|_{\mathrm{L}^2(\Gamma_1,\mathbb{R}^d)},$$

for all $w \in \mathrm{H}^{1/2}_{00}(\Gamma_1,\mathbb{R}^d)$, then v can be identified to an element $h \in \mathrm{L}^2(\Gamma_1,\mathbb{R}^d)$ with

$$\langle v, w \rangle_{\mathrm{H}^{-1/2}_{00}(\Gamma_1,\mathbb{R}^d) \times \mathrm{H}^{1/2}_{00}(\Gamma_1,\mathbb{R}^d)} = \langle h, w \rangle_{\mathrm{L}^2(\Gamma_1,\mathbb{R}^d)},$$

for all $w \in \mathrm{H}^{1/2}_{00}(\Gamma_1,\mathbb{R}^d)$.

The next proposition, known as divergence formula, can be found in [6, Theorem 4.4.7 p.104].

Theorem 2.16 (Divergence formula)**.** *Let $v \in \mathrm{H}_{\mathrm{div}}(\Omega,\mathbb{R}^{d\times d})$ where*

$$\mathrm{H}_{\mathrm{div}}(\Omega,\mathbb{R}^{d\times d}) := \left\{ w \in \mathrm{L}^2(\Omega,\mathbb{R}^{d\times d}) \mid \mathrm{div}(w) \in \mathrm{L}^2(\Omega,\mathbb{R}^d) \right\},$$

and where $\mathrm{div}(v)$ is the vector whose the i-th component is defined by $\mathrm{div}(v)_i := \mathrm{div}(v_i) \in \mathrm{L}^2(\Omega,\mathbb{R})$, where $v_i \in \mathrm{L}^2(\Omega,\mathbb{R}^d)$ is the i-th line of v for all $i \in [[1,d]]$. Then v admits a normal trace, denoted by $v\mathrm{n} \in \mathrm{H}^{-1/2}(\Gamma,\mathbb{R}^d)$, satisfying

$$\int_\Omega \mathrm{div}(v) \cdot w + \int_\Omega v : \nabla w = \langle v\mathrm{n}, w \rangle_{\mathrm{H}^{-1/2}(\Gamma,\mathbb{R}^d) \times \mathrm{H}^{1/2}(\Gamma,\mathbb{R}^d)}, \qquad \forall w \in \mathrm{H}^1(\Omega,\mathbb{R}^d).$$

3 Main result

In this section let $d \in \{2,3\}$ and Ω be a nonempty bounded connected open subset of $\mathbb{R}^d$ with a $\mathcal{C}^1$-boundary denoted by $\Gamma := \partial\Omega$ (see Remark 3.17 for comments on this $\mathcal{C}^1$-regularity assumption). We consider the decomposition

$$\Gamma =: \Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{N}},$$

where Γ_{D} and Γ_{N} are two measurable (with positive measure) pairwise disjoint subsets of Γ such that almost every point of Γ_{N} belongs to $\mathrm{int}_\Gamma(\Gamma_{\mathrm{N}})$ (see Remark 3.15 for comments on this last assumption). We introduce $\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ the linear subspace of $\mathrm{H}^1(\Omega,\mathbb{R}^d)$ defined by

$$\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d) := \left\{ w \in \mathrm{H}^1(\Omega,\mathbb{R}^d) \mid w = 0 \textit{ a.e. } \text{on } \Gamma_{\mathrm{D}} \right\}.$$

Moreover, we assume that Ω is an elastic solid satisfying the linear elastic model (see, e.g., [28]), that is

$$\sigma(w) = \mathrm{A}\mathrm{e}(w),$$

where σ is the Cauchy stress tensor, A the stiffness tensor, and e is the infinitesimal strain tensor defined by

$$\mathrm{e}(w) := \frac{1}{2}(\nabla w + \nabla w^\top),$$

for all displacement field $w \in \mathrm{H}^1(\Omega,\mathbb{R}^d)$. We also assume that all coefficients of A are measurable (denoted by a_{ijkl} for all $(i,j,k,l) \in \{1,...,d\}^4$) and that there exist two constants $\alpha > 0$ and $\gamma > 0$ such that all coefficients of A and e (denoted by ϵ_{ij} for all $(i,j) \in \{1,...,d\}^2$) satisfy

$$a_{ijkl}(x) = a_{jikl}(x) = a_{lkij}(x), \qquad |a_{ijkl}(x)| \le \alpha,$$

and also

$$\sum_{i=1}^{d}\sum_{j=1}^{d}\sum_{k=1}^{d}\sum_{l=1}^{d} a_{ijkl}\epsilon_{ij}(w_1)(x)\epsilon_{kl}(w_2)(x) \ge \gamma \sum_{i=1}^{d}\sum_{j=1}^{d} \epsilon_{ij}(w_1)(x)\epsilon_{ij}(w_2)(x),$$

for all displacement field $w_1, w_2 \in \mathrm{H}^1(\Omega,\mathbb{R}^d)$ and for almost all $x \in \Omega$. Moreover, since Γ_{D} has a positive measure, then we can deduce that

$$\begin{array}{rccl} \langle \cdot,\cdot \rangle_{\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)} : & \left(\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)\right)^2 & \longrightarrow & \mathbb{R} \\ & (w_1,w_2) & \longmapsto & \displaystyle\int_\Omega \mathrm{A}\mathrm{e}(w_1) : \mathrm{e}(w_2), \end{array}$$

is a scalar product on $\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ (see, e.g., [15, Chapter 3]) and we denote by $\|\cdot\|_{\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)}$ the corresponding norm. Moreover, from the assumptions on A, note that $\mathrm{A}\mathrm{e}(w) = \mathrm{A}\nabla w$ for all $w \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$.

We denote by $\mathrm{n} \in \mathcal{C}^0(\Gamma)$ the outward-pointing unit normal vector to Γ. Therefore, for any $w \in \mathrm{L}^2(\Gamma,\mathbb{R}^d)$, one has $w = w_{\mathrm{n}}\mathrm{n} + w_\tau$, where $w_{\mathrm{n}} := w \cdot \mathrm{n} \in \mathrm{L}^2(\Gamma,\mathbb{R})$ and $w_\tau := w - w_{\mathrm{n}}\mathrm{n} \in \mathrm{L}^2(\Gamma,\mathbb{R}^d)$. In particular, if the stress vector $\mathrm{A}\mathrm{e}(w)\mathrm{n}$ is in $\mathrm{L}^2(\Gamma_{\mathrm{N}},\mathbb{R}^d)$ for some $w \in \mathrm{H}^1(\Omega,\mathbb{R}^d)$, then we use the notation

$$\mathrm{A}\mathrm{e}(w)\mathrm{n} = \sigma_{\mathrm{n}}(w)\mathrm{n} + \sigma_\tau(w),$$

where $\sigma_{\mathrm{n}}(w) \in \mathrm{L}^2(\Gamma_{\mathrm{N}},\mathbb{R})$ is the normal stress and $\sigma_\tau(w) \in \mathrm{L}^2(\Gamma_{\mathrm{N}},\mathbb{R}^d)$ is the shear stress. We also denote by $\|\cdot\|$ the Euclidean norm on $\mathbb{R}^d$ and, for all $w \in \mathrm{L}^2(\Gamma,\mathbb{R}^d)$, $\|w_\tau\| \in \mathrm{L}^2(\Gamma)$ is defined by

$$\begin{array}{rccl} \|w_\tau\| : & \Gamma & \longrightarrow & \mathbb{R} \\ & s & \longmapsto & \|w_\tau(s)\|. \end{array}$$

The rest of this section is organized as follows. Subsection 3.1 introduces three general boundary value problems involved all along the paper: a Dirichlet-Neumann problem (see Problem (DN)), a tangential Signorini problem (see Problem (SP)) and a Tresca friction problem (see Problem TP). In Subsection 3.2, the sensitivity analysis of the Tresca friction problem is performed and we establish the main result of this paper (see Theorem 3.23).

3.1 Three general boundary value problems

For the needs of this subsection, let us fix $f \in \mathrm{L}^2(\Omega, \mathbb{R}^d)$. Only the proofs of Subsection 3.1.2 are detailed since the tangential Signorini problem is, to the best of our knowledge, new in the literature. For the proofs of the other problems, they are classical and close to the ones presented in [3] and thus they are left to the reader.

3.1.1 A general problem with Dirichlet-Neumann conditions

Let $z \in \mathrm{L}^2(\Gamma_\mathrm{N}, \mathbb{R}^d)$ and consider the general Dirichlet-Neumann problem given by

$$\begin{cases} -\mathrm{div}(\mathrm{Ae}(F)) = f & \text{in } \Omega, \\ F = 0 & \text{on } \Gamma_\mathrm{D}, \\ \mathrm{Ae}(F)\mathrm{n} = z & \text{on } \Gamma_\mathrm{N}. \end{cases} \tag{DN}$$

Definition 3.1 (Strong solution to the Dirichlet-Neumann problem)**.** *A (strong) solution to the Dirichlet-Neumann problem* (DN) *is a function* $F \in \mathrm{H}^1(\Omega, \mathbb{R}^d)$ *such that* $-\mathrm{div}(\mathrm{Ae}(F)) = f$ *in* $\mathcal{D}'(\Omega, \mathbb{R}^d)$*,* $F = 0$ *a.e. on* Γ_D*,* $\mathrm{Ae}(F)\mathrm{n} \in \mathrm{L}^2(\Gamma_\mathrm{N}, \mathbb{R}^d)$ *with* $\mathrm{Ae}(F)\mathrm{n} = z$ *a.e. on* Γ_N*.*

Definition 3.2 (Weak solution to the Dirichlet-Neumann problem)**.** *A weak solution to the Dirichlet-Neumann problem* (DN) *is a function* $F \in \mathrm{H}^1_\mathrm{D}(\Omega, \mathbb{R}^d)$ *such that*

$$\int_\Omega \mathrm{Ae}(F) : \mathrm{e}(w) = \int_\Omega f \cdot w + \int_{\Gamma_\mathrm{N}} z \cdot w, \qquad \forall w \in \mathrm{H}^1_\mathrm{D}(\Omega, \mathbb{R}^d).$$

Proposition 3.3. *A function* $F \in \mathrm{H}^1(\Omega, \mathbb{R}^d)$ *is a (strong) solution to the Dirichlet-Neumann problem* (DN) *if and only if* F *is a weak solution to the Dirichlet-Neumann problem* (DN)*.*

Using the Riesz representation theorem, we obtain the following existence/uniqueness result.

Proposition 3.4. *The Dirichlet-Neumann problem* (DN) *admits a unique solution* $F \in \mathrm{H}^1_\mathrm{D}(\Omega, \mathbb{R}^d)$*. Moreover there exists a constant* $C \geq 0$ *(depending only on* Ω*) such that*

$$\|F\|_{\mathrm{H}^1_\mathrm{D}(\Omega,\mathbb{R}^d)} \leq C \left(\|f\|_{\mathrm{L}^2(\Omega,\mathbb{R}^d)} + \|z\|_{\mathrm{L}^2(\Gamma_\mathrm{N},\mathbb{R}^d)} \right).$$

3.1.2 A general tangential Signorini problem

In this part we assume that Γ_N is decomposed (up to a null set) as

$$\Gamma_\mathrm{N} =: \Gamma_{\mathrm{N_T}} \cup \Gamma_{\mathrm{N_R}} \cup \Gamma_{\mathrm{N_S}},$$

where $\Gamma_{\mathrm{N_T}}$, $\Gamma_{\mathrm{N_R}}$, $\Gamma_{\mathrm{N_S}}$ are three measurable pairwise disjoint subsets of Γ_N. Moreover let $h \in \mathrm{L}^2(\Gamma_\mathrm{N})$, $\ell \in \mathrm{L}^2(\Gamma_{\mathrm{N_R}} \cup \Gamma_{\mathrm{N_S}})$, $v \in \mathrm{L}^\infty(\Gamma_{\mathrm{N_R}} \cup \Gamma_{\mathrm{N_S}}, \mathbb{R}^d)$ such that $||v||_{\mathrm{L}^\infty(\Gamma_{\mathrm{N_R}} \cup \Gamma_{\mathrm{N_S}}, \mathbb{R}^d)} \leq 1$, $k \in \mathrm{L}^4(\Gamma_{\mathrm{N_R}})$ such that $k > 0$ *a.e.* on $\Gamma_{\mathrm{N_R}}$, and we denote, for almost all $s \in \Gamma_{\mathrm{N_S}}$, $\mathbb{R}_- v_\tau(s) := \{y \in \mathbb{R}^d \mid \exists \nu \leq 0 \text{ such that } y = \nu v_\tau(s)\}$. The general tangential Signorini problem is given by

$$
\begin{cases}
-\mathrm{div}(\mathrm{Ae}(u)) = f & \text{in } \Omega, \\
u = 0 & \text{on } \Gamma_{\mathrm{D}}, \\
\sigma_{\mathrm{n}}(u) = h & \text{on } \Gamma_{\mathrm{N}}, \\
u_\tau = 0 & \text{on } \Gamma_{\mathrm{N_T}}, \\
\sigma_\tau(u) + k\,(u_\tau - (u_\tau \cdot v_\tau)\,v_\tau) = \ell v_\tau & \text{on } \Gamma_{\mathrm{N_R}}, \\
u_\tau \in \mathbb{R}_- v_\tau, (\sigma_\tau(u) - \ell v_\tau)\cdot v_\tau \leq 0 \text{ and } u_\tau \cdot (\sigma_\tau(u) - \ell v_\tau) = 0 & \text{on } \Gamma_{\mathrm{N_S}}.
\end{cases} \tag{SP}
$$

Definition 3.5 (Strong solution to the tangential Signorini problem)**.** *A (strong) solution to the tangential Signorini problem* (SP) *is a function* $u \in \mathrm{H}^1(\Omega, \mathbb{R}^d)$ *such that* $-\mathrm{div}(\mathrm{Ae}(u)) = f$ *in* $\mathcal{D}'(\Omega, \mathbb{R}^d)$*,* $u = 0$ *a.e. on* Γ_{D}*,* $u_\tau = 0$ *a.e. on* $\Gamma_{\mathrm{N_T}}$*,* $\mathrm{Ae}(u)\mathrm{n} \in \mathrm{L}^2(\Gamma_{\mathrm{N}}, \mathbb{R}^d)$ *with* $\sigma_{\mathrm{n}}(u) = h$ *a.e. on* Γ_{N}*,* $\sigma_\tau(u) + k\,(u_\tau - (u_\tau \cdot v_\tau) v_\tau) = \ell v_\tau$ *a.e. on* $\Gamma_{\mathrm{N_R}}$*,* $u_\tau \in \mathbb{R}_- v_\tau, (\sigma_\tau(u) - \ell v_\tau)\cdot v_\tau \leq 0$ *and* $u_\tau \cdot (\sigma_\tau(u) - \ell v_\tau) = 0$ *a.e. on* $\Gamma_{\mathrm{N_S}}$*.*

Definition 3.6 (Weak solution to the tangential Signorini problem)**.** *A weak solution to the tangential Signorini problem* (SP) *is a function* $u \in \mathcal{K}^1(\Omega, \mathbb{R}^d)$ *such that*

$$
\int_\Omega \mathrm{Ae}(u) : \mathrm{e}(w-u) \geq \int_\Omega f\cdot(w-u) + \int_{\Gamma_{\mathrm{N}}} h(w_{\mathrm{n}} - u_{\mathrm{n}}) + \int_{\Gamma_{\mathrm{N_R}}} (\ell v_\tau - k\,(u_\tau - (u_\tau\cdot v_\tau)v_\tau))\cdot(w_\tau - u_\tau)
$$
$$
+ \int_{\Gamma_{\mathrm{N_S}}} \ell v_\tau \cdot (w_\tau - u_\tau), \qquad \forall w \in \mathcal{K}^1(\Omega, \mathbb{R}^d), \tag{3.1}
$$

where $\mathcal{K}^1(\Omega, \mathbb{R}^d)$ *is the nonempty closed convex subset of* $\mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ *given by*

$$
\mathcal{K}^1(\Omega, \mathbb{R}^d) := \left\{ w \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d) \mid w_\tau = 0 \text{ a.e. on } \Gamma_{\mathrm{N_T}} \text{ and } w_\tau \in \mathbb{R}_- v_\tau \text{ a.e. on } \Gamma_{\mathrm{N_S}} \right\}.
$$

One can easily prove that a (strong) solution to the tangential Signorini problem (SP) is also a weak solution. However, to the best of our knowledge, without additional assumptions one cannot prove the converse. To get the equivalence, one can assume, in particular, that the decomposition $\Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{N_T}} \cup \Gamma_{\mathrm{N_R}} \cup \Gamma_{\mathrm{N_S}}$ of Γ is *consistent* in the following sense.

Definition 3.7 (Consistent decomposition)**.** *The decomposition* $\Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{N_T}} \cup \Gamma_{\mathrm{N_R}} \cup \Gamma_{\mathrm{N_S}}$ *of* Γ *is said to be* consistent *if:*

(i) *for almost all* $s \in \Gamma_{\mathrm{N_S}}$*,* $s \in \mathrm{int}_\Gamma(\Gamma_{\mathrm{N_S}})$*;*

(ii) *the nonempty closed convex subset* $\mathcal{K}^{1/2}(\Gamma, \mathbb{R}^d)$ *of* $\mathrm{H}^{1/2}(\Gamma, \mathbb{R}^d)$ *defined by*

$$
\mathcal{K}^{1/2}(\Gamma, \mathbb{R}^d) := \Big\{ w \in \mathrm{H}^{1/2}(\Gamma, \mathbb{R}^d) \mid w = 0 \text{ a.e. on } \Gamma_{\mathrm{D}},\ w_\tau = 0 \text{ a.e. on } \Gamma_{\mathrm{N_T}} \text{ and } w_\tau \in \mathbb{R}_- v_\tau \text{ a.e. on } \Gamma_{\mathrm{N_S}} \Big\},
$$

is dense in the nonempty closed convex subset $\mathcal{K}^0(\Gamma, \mathbb{R}^d)$ *of* $\mathrm{L}^2(\Gamma, \mathbb{R}^d)$ *given by*

$$
\mathcal{K}^0(\Gamma, \mathbb{R}^d) := \Big\{ w \in \mathrm{L}^2(\Gamma, \mathbb{R}^d) \mid w = 0 \text{ a.e. on } \Gamma_{\mathrm{D}},\ w_\tau = 0 \text{ a.e. on } \Gamma_{\mathrm{N_T}} \text{ and } w_\tau \in \mathbb{R}_- v_\tau \text{ a.e. on } \Gamma_{\mathrm{N_S}} \Big\}.
$$

Proposition 3.8. *Let* $u \in \mathrm{H}^1(\Omega, \mathbb{R}^d)$.

(i) *If* u *is a (strong) solution to the tangential Signorini problem* (SP)*, then* u *is a weak solution to the Signorini problem* (SP)*.*

(ii) *If* u *is a weak solution to the tangential Signorini problem* (SP) *such that* $\mathrm{Ae}(u)\mathrm{n} \in \mathrm{L}^2(\Gamma_\mathrm{N}, \mathbb{R}^d)$ *and the decomposition* $\Gamma_\mathrm{D} \cup \Gamma_{\mathrm{N_T}} \cup \Gamma_{\mathrm{N_R}} \cup \Gamma_{\mathrm{N_S}}$ *of* Γ *is consistent, then* u *is a (strong) solution to the tangential Signorini problem* (SP)*.*

Proof. (i) Assume that u is a (strong) solution to the tangential Signorini problem (SP). Then, from the boundary conditions, $u \in \mathcal{K}^1(\Omega, \mathbb{R}^d)$. Moreover, since $-\mathrm{div}(\mathrm{Ae}(u)) = f$ in $\mathcal{D}'(\Omega, \mathbb{R}^d)$ and $f \in \mathrm{L}^2(\Omega, \mathbb{R}^d)$, then $-\mathrm{div}(\mathrm{Ae}(u)) = f$ in $\mathrm{L}^2(\Omega, \mathbb{R}^d)$. Hence, from divergence formula (see Proposition 2.16), one gets

$$\int_\Omega \mathrm{Ae}(u) : \mathrm{e}(w-u) - \langle \mathrm{Ae}(u)\mathrm{n}, w-u \rangle_{\mathrm{H}^{-1/2}(\Gamma,\mathbb{R}^d)\times \mathrm{H}^{1/2}(\Gamma,\mathbb{R}^d)} = \int_\Omega f \cdot (w-u),$$

for all $w \in \mathcal{K}^1(\Omega, \mathbb{R}^d)$. Moreover, for all $w \in \mathcal{K}^1(\Omega, \mathbb{R}^d)$, $w \in \mathrm{H}^{1/2}_{00}(\Gamma_\mathrm{N}, \mathbb{R}^d)$ which can be identified to a linear subspace of $\mathrm{H}^{1/2}(\Gamma, \mathbb{R}^d)$, hence

$$\int_\Omega \mathrm{Ae}(u) : \mathrm{e}(w-u) - \langle \mathrm{Ae}(u)\mathrm{n}, w-u \rangle_{\mathrm{H}^{-1/2}_{00}(\Gamma_\mathrm{N},\mathbb{R}^d)\times \mathrm{H}^{1/2}_{00}(\Gamma_\mathrm{N},\mathbb{R}^d)} = \int_\Omega f \cdot (w-u),$$

for all $w \in \mathcal{K}^1(\Omega, \mathbb{R}^d)$. Furthermore, since $\mathrm{Ae}(u)\mathrm{n} \in \mathrm{L}^2(\Gamma_\mathrm{N}, \mathbb{R}^d)$, it follows that

$$\langle \mathrm{Ae}(u)\mathrm{n}, w-u \rangle_{\mathrm{H}^{-1/2}_{00}(\Gamma_\mathrm{N},\mathbb{R}^d)\times \mathrm{H}^{1/2}_{00}(\Gamma_\mathrm{N},\mathbb{R}^d)} = \int_{\Gamma_\mathrm{N}} \mathrm{Ae}(u)\mathrm{n} \cdot (w-u),$$

for all $w \in \mathcal{K}^1(\Omega, \mathbb{R}^d)$. Using the decomposition of $\mathrm{Ae}(u)\mathrm{n}$ on its tangential and normal components, one has

$$\int_{\Gamma_\mathrm{N}} \mathrm{Ae}(u)\mathrm{n} \cdot (w-u) = \int_{\Gamma_\mathrm{N}} \sigma_\mathrm{n}(u)(w_\mathrm{n} - u_\mathrm{n}) + \int_{\Gamma_{\mathrm{N_R}} \cup \Gamma_{\mathrm{N_S}}} \sigma_\tau(u) \cdot (w_\tau - u_\tau),$$

for all $w \in \mathcal{K}^1(\Omega, \mathbb{R}^d)$. From the boundary conditions, one has $\sigma_\mathrm{n}(u) = h$ *a.e.* on Γ_N and $\sigma_\tau(u) = \ell v_\tau - k\,(u_\tau - (u_\tau \cdot v_\tau)v_\tau)$ *a.e.* on $\Gamma_{\mathrm{N_R}}$. Moreover one has

$$\sigma_\tau(u) \cdot (w_\tau - u_\tau) = \sigma_\tau(u) \cdot w_\tau - \sigma_\tau(u) \cdot u_\tau \geq \ell v_\tau \cdot w_\tau - \ell v_\tau \cdot u_\tau = \ell v_\tau \cdot (w_\tau - u_\tau)\,,$$

a.e. on $\Gamma_{\mathrm{N_S}}$. This concludes the proof of the first item.

(ii) Assume that u is a weak solution to the tangential Signorini problem (SP). Then $u \in \mathcal{K}^1(\Omega, \mathbb{R}^d)$. For all $\varphi \in \mathcal{D}(\Omega, \mathbb{R}^d)$, considering $w := u \pm \varphi \in \mathcal{K}^1(\Omega, \mathbb{R}^d)$ in Inequality (3.1), one gets $-\mathrm{div}(\mathrm{Ae}(u)) = f$ in $\mathcal{D}'(\Omega, \mathbb{R}^d)$, then also in $\mathrm{L}^2(\Omega, \mathbb{R}^d)$ since $f \in \mathrm{L}^2(\Omega, \mathbb{R}^d)$. Hence we can apply the divergence formula (see Proposition 2.16) in Inequality (3.1) to get that

$$\begin{aligned} \langle \mathrm{Ae}(u)\mathrm{n}, w-u \rangle_{\mathrm{H}^{-1/2}(\Gamma,\mathbb{R}^d)\times \mathrm{H}^{1/2}(\Gamma,\mathbb{R}^d)} &\geq \int_{\Gamma_\mathrm{N}} h(w_\mathrm{n} - u_\mathrm{n}) \\ &+ \int_{\Gamma_{\mathrm{N_R}}} (\ell v_\tau - k\,(u_\tau - (u_\tau \cdot v_\tau)v_\tau)) \cdot (w_\tau - u_\tau) + \int_{\Gamma_{\mathrm{N_S}}} \ell v_\tau \cdot (w_\tau - u_\tau), \end{aligned}$$

for all $w \in \mathcal{K}^1(\Omega, \mathbb{R}^d)$. Moreover, similarly to (i) and from the assumption $\mathrm{Ae}(u)\mathrm{n} \in \mathrm{L}^2(\Gamma_\mathrm{N}, \mathbb{R}^d)$, one gets

$$\int_{\Gamma_{\mathrm{N}}} \sigma_{\mathrm{n}}(u)(w_{\mathrm{n}}-u_{\mathrm{n}}) + \int_{\Gamma_{\mathrm{N_R}}\cup\Gamma_{\mathrm{N_S}}} \sigma_\tau(u)\cdot(w_\tau-u_\tau) \geq \int_{\Gamma_{\mathrm{N}}} h(w_{\mathrm{n}}-u_{\mathrm{n}}) \\ + \int_{\Gamma_{\mathrm{N_R}}} (\ell v_\tau - k\,(u_\tau-(u_\tau\cdot v_\tau)v_\tau))\cdot(w_\tau-u_\tau) + \int_{\Gamma_{\mathrm{N_S}}} \ell v_\tau\cdot(w_\tau-u_\tau), \tag{3.2}$$

for all $w\in\mathcal{K}^1(\Omega,\mathbb{R}^d)$, then also for all $w\in\mathcal{K}^{1/2}(\Gamma,\mathbb{R}^d)$. From the assumption that the decomposition $\Gamma_{\mathrm{D}}\cup\Gamma_{\mathrm{N_T}}\cup\Gamma_{\mathrm{N_R}}\cup\Gamma_{\mathrm{N_S}}$ of Γ is consistent, $\mathcal{K}^{1/2}(\Gamma,\mathbb{R}^d)$ is dense in $\mathcal{K}^0(\Gamma,\mathbb{R}^d)$. Therefore, since $k\in\mathrm{L}^4(\Gamma_{\mathrm{N_R}})$, $||v||_{\mathrm{L}^\infty(\Gamma_{\mathrm{N_R}}\cup\Gamma_{\mathrm{N_S}},\mathbb{R}^d)}\leq 1$ and from the continuous embedding $\mathrm{H}^1(\Omega,\mathbb{R}^d)\hookrightarrow\mathrm{L}^4(\Gamma,\mathbb{R}^d)$, we deduce that Inequality (3.2) is still true for all $w\in\mathcal{K}^0(\Gamma,\mathbb{R}^d)$.

By considering the function $w:=u\pm\psi\mathrm{n}\in\mathcal{K}^0(\Gamma,\mathbb{R}^d)$ in Inequality (3.2), where $\psi\in\mathrm{L}^2(\Gamma)$ is given by

$$\psi=\begin{cases} 0 & \text{on } \Gamma_{\mathrm{D}}, \\ \phi & \text{on } \Gamma_{\mathrm{N}}, \end{cases}$$

with ϕ any function in $\mathrm{L}^2(\Gamma_{\mathrm{N}})$, one deduces that $\sigma_{\mathrm{n}}(u)=h$ *a.e.* on Γ_{N}.

By considering $w:=u\pm w_\phi\in\mathcal{K}^0(\Gamma,\mathbb{R}^d)$ in Inequality (3.2), where $w_\phi\in\mathrm{L}^2(\Gamma,\mathbb{R}^d)$ is given by

$$w_\phi=\begin{cases} 0 & \text{on } \Gamma_{\mathrm{D}}\cup\Gamma_{\mathrm{N_T}}\cup\Gamma_{\mathrm{N_S}}, \\ \phi & \text{on } \Gamma_{\mathrm{N_R}}, \end{cases}$$

with ϕ any function in $\mathrm{L}^2(\Gamma_{\mathrm{N_R}},\mathbb{R}^d)$, one gets that $\sigma_\tau(u)=\ell v_\tau-k\,(u_\tau-(u_\tau\cdot v_\tau)v_\tau)$ *a.e.* on $\Gamma_{\mathrm{N_R}}$. Hence Inequality (3.2) becomes

$$\int_{\Gamma_{\mathrm{N_S}}} \sigma_\tau(u)\cdot(w_\tau-u_\tau) \geq \int_{\Gamma_{\mathrm{N_S}}} \ell v_\tau\cdot(w_\tau-u_\tau), \tag{3.3}$$

for all $w\in\mathcal{K}^0(\Gamma,\mathbb{R}^d)$. Let $s\in\Gamma_{\mathrm{N_S}}$ be a Lebesgue point of $\sigma_\tau(u)\cdot v_\tau\in\mathrm{L}^2(\Gamma_{\mathrm{N_R}}\cup\Gamma_{\mathrm{N_S}})$ and of $\ell\,\|v_\tau\|_2^2\in\mathrm{L}^2(\Gamma_{\mathrm{N_R}}\cup\Gamma_{\mathrm{N_S}})$, such that $s\in\mathrm{int}_\Gamma(\Gamma_{\mathrm{N_S}})$. By considering the function $w:=u-\psi v_\tau\in\mathcal{K}^0(\Gamma,\mathbb{R}^d)$ in Inequality (3.3), where $\psi\in\mathrm{L}^2(\Gamma)$ is defined by

$$\psi:=\begin{cases} 1 & \text{on } B_\Gamma(s,\varepsilon), \\ 0 & \text{on } \Gamma\backslash B_\Gamma(s,\varepsilon), \end{cases}$$

for $\varepsilon>0$ such that $B_\Gamma(s,\varepsilon)\subset\Gamma_{\mathrm{N_S}}$, one gets that

$$\frac{1}{|B_\Gamma(s,\varepsilon)|}\int_{B_\Gamma(s,\varepsilon)} \sigma_\tau(u)\cdot v_\tau \leq \frac{1}{|B_\Gamma(s,\varepsilon)|}\int_{B_\Gamma(s,\varepsilon)} \ell\,\|v_\tau\|_2^2,$$

and thus $(\sigma_\tau(u)(s)-\ell(s)v_\tau(s))\cdot v_\tau(s)\leq 0$ by letting $\varepsilon\to 0^+$. Moreover, since almost every point of $\Gamma_{\mathrm{N_S}}$ are in $\mathrm{int}_\Gamma(\Gamma_{\mathrm{N_S}})$ and are Lesbegue points of $\sigma_\tau(u)\cdot v_\tau\in\mathrm{L}^2(\Gamma_{\mathrm{N_R}}\cup\Gamma_{\mathrm{N_S}})$ and of $\ell\,\|v_\tau\|_2^2\in\mathrm{L}^2(\Gamma_{\mathrm{N_R}}\cup\Gamma_{\mathrm{N_S}})$, one deduces

$$(\sigma_\tau(u)-\ell v_\tau)\cdot v_\tau\leq 0,$$

a.e. on $\Gamma_{\mathrm{N_S}}$. Finally, by considering $w=0$ and $w=2u$ in Inequality (3.3), one gets

$$\int_{\Gamma_{\mathrm{N_S}}} u_\tau\cdot(\sigma_\tau(u)-\ell v_\tau)=0,$$

therefore $u_\tau\cdot(\sigma_\tau(u)-\ell v_\tau)=0$ *a.e.* on $\Gamma_{\mathrm{N_S}}$ since $u\in\mathcal{K}^1(\Omega,\mathbb{R}^d)$. The proof is complete. □

Now let us prove that there exists a unique solution to the tangential Signorini problem (SP). To this aim let us introduce the functional Ψ defined by

$$\begin{array}{rrcl} \Psi: & \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d) & \longrightarrow & \mathbb{R} \\ & w & \longmapsto & \Psi(w) := \displaystyle\int_{\Gamma_{\mathrm{N_R}}} \frac{k}{2}\left(\|w_\tau\|^2 - |w_\tau\cdot v_\tau|^2\right). \end{array}$$

Note that Ψ is well defined since $k \in \mathrm{L}^4(\Gamma_{\mathrm{N_R}})$, $||v|| \leq 1$ *a.e.* on $\Gamma_{\mathrm{N_R}}$ and from the continuous embedding $\mathrm{H}^1(\Omega,\mathbb{R}^d)\hookrightarrow\mathrm{L}^4(\Gamma,\mathbb{R}^d)$.

Lemma 3.9. *The functional Ψ is convex and Fréchet differentiable on $\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ and, for all $w_0 \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$, $\nabla\Psi(w_0) \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ is the unique solution to the Dirichlet-Neumann problem*

$$\left\{\begin{aligned} -\mathrm{div}(\mathrm{Ae}(\nabla\Psi(w_0))) &= 0 && \text{in } \Omega, \\ \nabla\Psi(w_0) &= 0 && \text{on } \Gamma_{\mathrm{D}}, \\ \mathrm{Ae}(\nabla\Psi(w_0))\mathrm{n} &= 0 && \text{on } \Gamma_{\mathrm{N_T}}\cup\Gamma_{\mathrm{N_S}}, \\ \sigma_{\mathrm{n}}(\nabla\Psi(w_0)) &= 0 && \text{on } \Gamma_{\mathrm{N_R}}, \\ \sigma_\tau(\nabla\Psi(w_0)) &= k\left(w_{0_\tau} - (w_{0_\tau}\cdot v_\tau)\, v_\tau\right) && \text{on } \Gamma_{\mathrm{N_R}}. \end{aligned}\right. \tag{3.4}$$

Proof. Let us start with the convexity of Ψ. Take $w_1, w_2 \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ and $\nu \in (0,1)$. Then

$$\Psi(\nu w_1 + (1-\nu)w_2) - \nu\Psi(w_1) - (1-\nu)\Psi(w_2) =$$

$$\int_{\Gamma_{\mathrm{N_R}}} -\frac{k}{2}\nu(1-\nu)[\Big[\|w_{1_\tau}\|^2 + \|w_{2_\tau}\|^2 + 2w_{1_\tau}\cdot w_{2_\tau} - |w_{1_\tau}\cdot v_\tau|^2 - |w_{2_\tau}\cdot v_\tau|^2 - 2(w_{1_\tau}\cdot v_\tau)(w_{2_\tau}\cdot v_\tau)\Big]$$

$$= \int_{\Gamma_{\mathrm{N_R}}} -\frac{k}{2}\nu(1-\nu)\,\|w_{1_\tau} + w_{2_\tau}\|^2 + \int_{\Gamma_{\mathrm{N_R}}} \frac{k}{2}\nu(1-\nu)\,|(w_{1_\tau} + w_{2_\tau})\cdot v_\tau|^2.$$

Since $k > 0$ and $\|v\| \leq 1$ *a.e.* on $\Gamma_{\mathrm{N_R}}$, one deduces

$$\Psi(\nu w_1 + (1-\nu)w_2) - \nu\Psi(w_1) - (1-\nu)\Psi(w_2) \leq$$

$$\int_{\Gamma_{\mathrm{N_R}}} -\frac{k}{2}\nu(1-\nu)\,\|w_{1_\tau} + w_{2_\tau}\|^2 + \int_{\Gamma_{\mathrm{N_R}}} \frac{k}{2}\nu(1-\nu)\,\|w_{1_\tau} + w_{2_\tau}\|^2\,\|v_\tau\|^2 \leq 0.$$

Thus Ψ is convex on $\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$. Now let us prove that Ψ is Fréchet differentiable. For $w_0 \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ and $w \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$, it holds that

$$\Psi(w_0 + w) = \Psi(w_0) + \int_{\Gamma_{\mathrm{N_R}}} k\left(w_{0_\tau} - (w_{0_\tau}\cdot v_\tau)\, v_\tau\right)\cdot w_\tau + \int_{\Gamma_{\mathrm{N_R}}} \frac{k}{2}\left(\|w_\tau\|^2 - |w_\tau\cdot v_\tau|^2\right).$$

Moreover one has

$$\int_{\Gamma_{\mathrm{N_R}}} \frac{k}{2}\left(\|w_\tau\|^2 - |w_\tau\cdot v_\tau|^2\right) = o(w),$$

where o stands for the standard Bachmann-Landau notation for the $\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$-norm. Moreover the map

$$w \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d) \mapsto \int_{\Gamma_{\mathrm{N_R}}} k\left(w_{0_\tau} - (w_{0_\tau}\cdot v_\tau)\, v_\tau\right)\cdot w_\tau \in \mathbb{R},$$

is linear and continuous. Therefore Ψ is Fréchet differentiable in $w_0 \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ and

$$\langle \nabla\Psi(w_0), w\rangle_{\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)} = \int_{\Gamma_{\mathrm{N_R}}} k\left(w_{0_\tau} - (w_{0_\tau}\cdot v_\tau)\, v_\tau\right)\cdot w_\tau, \qquad \forall w \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d).$$

In other words $\nabla\Psi(w_0) \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ is the unique solution to the Dirichlet-Neumann problem (3.4). The proof is complete. □

Proposition 3.10. *The tangential Signorini problem* (SP) *admits a unique weak solution $u \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ which is given by*

$$u = \mathrm{prox}_{\Psi+\iota_{\mathcal{K}^1(\Omega,\mathbb{R}^d)}}(F),$$

where $F \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ is the unique solution to the Dirichlet-Neumann problem (DN) *with $z := h\mathrm{n} + \ell v_\tau \in \mathrm{L}^2(\Gamma_{\mathrm{N}},\mathbb{R}^d)$, and $\mathrm{prox}_{\Psi+\iota_{\mathcal{K}^1(\Omega,\mathbb{R}^d)}}$ stands for the proximal operator associated with the functional $\Psi + \iota_{\mathcal{K}^1(\Omega,\mathbb{R}^d)}$.*

Proof. Let $F \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ be the solution to the Dirichlet-Neumann problem (DN) with $z := h\mathrm{n} + \ell v_\tau \in \mathrm{L}^2(\Gamma_{\mathrm{N}},\mathbb{R}^d)$. Then

$$\langle F, w\rangle_{\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)} = \int_\Omega f\cdot w + \int_{\Gamma_{\mathrm{N}}} h w_{\mathrm{n}} + \int_{\Gamma_{\mathrm{N}}} \ell v_\tau \cdot w_\tau, \qquad \forall w \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d).$$

Let $u \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ and note that $\Psi + \iota_{\mathcal{K}^1(\Omega,\mathbb{R}^d)}$ is a proper lower semi-continuous convex function on $\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$. Then u is the weak solution to the tangentiel Signorini problem (SP) if and only if $u \in \mathcal{K}^1(\Omega,\mathbb{R}^d)$ and

$$\begin{aligned}\langle u, w-u\rangle_{\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)} \geq{}& \int_\Omega f\cdot(w-u) + \int_{\Gamma_{\mathrm{N}}} h\,(w_{\mathrm{n}} - u_{\mathrm{n}})\\ &+ \int_{\Gamma_{\mathrm{N_R}}} \left(\ell v_\tau - k\left(u_\tau - (u_\tau\cdot v_\tau)\, v_\tau\right)\right)\cdot(w_\tau - u_\tau) + \int_{\Gamma_{\mathrm{N_S}}} \ell v_\tau\cdot(w_\tau - u_\tau), \qquad \forall w \in \mathcal{K}^1(\Omega,\mathbb{R}^d),\end{aligned}$$

i.e. if and only if

$$\int_{\Gamma_{\mathrm{N_R}}} k\left(u_\tau - (u_\tau\cdot v_\tau)\, v_\tau\right)\cdot(w_\tau - u_\tau) \geq \langle F-u, w-u\rangle_{\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)}, \qquad \forall w \in \mathcal{K}^1(\Omega,\mathbb{R}^d),$$

i.e. if and only if (see Proposition 2.4)

$$\Psi(w) - \Psi(u) \geq \langle F-u, w-u\rangle_{\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)}, \qquad \forall w \in \mathcal{K}^1(\Omega,\mathbb{R}^d),$$

i.e. if and only if

$$\langle F-u, w-u\rangle_{\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)} \leq \Psi(w) - \Psi(u) + \iota_{\mathcal{K}^1(\Omega,\mathbb{R}^d)}(w) - \iota_{\mathcal{K}^1(\Omega,\mathbb{R}^d)}(u), \qquad \forall w \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d),$$

i.e. if and only if $F - u \in \partial\big(\Psi + \iota_{\mathcal{K}^1(\Omega,\mathbb{R}^d)}\big)(u)$, i.e. if and only if $u = \mathrm{prox}_{\Psi+\iota_{\mathcal{K}^1(\Omega,\mathbb{R}^d)}}(F)$, which concludes the proof. □

3.1.3 A general Tresca friction problem

Let $h \in \mathrm{L}^2(\Gamma_{\mathrm{N}})$ and $g \in \mathrm{L}^2(\Gamma_{\mathrm{N}})$ such that $g > 0$ *a.e.* on Γ_{N}. Consider the general Tresca friction problem given by

$$\begin{cases} -\mathrm{div}(\mathrm{Ae}(u)) = f & \text{in } \Omega,\\ u = 0 & \text{on } \Gamma_{\mathrm{D}},\\ \sigma_{\mathrm{n}}(u) = h & \text{on } \Gamma_{\mathrm{N}},\\ \|\sigma_\tau(u)\| \leq g \text{ and } u_\tau\cdot\sigma_\tau(u) + g\,\|u_\tau\| = 0 & \text{on } \Gamma_{\mathrm{N}}.\end{cases} \tag{TP}$$

Definition 3.11 (Strong solution to the Tresca friction problem)**.** *A (strong) solution to the Tresca friction problem* (TP) *is a function* $u \in \mathrm{H}^1(\Omega,\mathbb{R}^d)$ *such that* $-\mathrm{div}(\mathrm{Ae}(u)) = f$ *in* $\mathcal{D}'(\Omega,\mathbb{R}^d)$*,* $u = 0$ *a.e. on* Γ_D*,* $\mathrm{Ae}(u)\mathrm{n} \in \mathrm{L}^2(\Gamma_\mathrm{N},\mathbb{R}^d)$ *with* $\sigma_\mathrm{n}(u) = h$*,* $\|\sigma_\tau(u)\| \leq g$ *and* $u_\tau \cdot \sigma_\tau(u) + g\,\|u_\tau\| = 0$ *a.e. on* Γ_N*.*

Definition 3.12 (Weak solution to the Tresca friction problem)**.** *A weak solution to the Tresca friction problem* (TP) *is a function* $u \in \mathrm{H}^1_\mathrm{D}(\Omega,\mathbb{R}^d)$ *such that*

$$\int_\Omega \mathrm{Ae}(u) : \mathrm{e}(w-u) + \int_{\Gamma_\mathrm{N}} g\,\|w_\tau\| - \int_{\Gamma_\mathrm{N}} g\,\|u_\tau\| \geq \int_\Omega f\cdot(w-u) + \int_{\Gamma_\mathrm{N}} h\,(w_\mathrm{n}-u_\mathrm{n})\,, \qquad \forall w \in \mathrm{H}^1_\mathrm{D}(\Omega,\mathbb{R}^d).$$

Proposition 3.13. *A function* $u \in \mathrm{H}^1(\Omega,\mathbb{R}^d)$ *is a (strong) solution to the Tresca friction problem* (TP) *if and only if* u *is a weak solution to the Tresca friction problem* (TP)*.*

From definition of the proximal operator (see Definition 2.3), one deduces the following existence/uniqueness result.

Proposition 3.14. *The Tresca friction problem* (TP) *admits a unique solution* $u \in \mathrm{H}^1_\mathrm{D}(\Omega,\mathbb{R}^d)$ *given by*

$$u = \mathrm{prox}_\phi(F),$$

where $F \in \mathrm{H}^1_\mathrm{D}(\Omega,\mathbb{R}^d)$ *is the solution to the Dirichlet-Neumann problem* (DN) *with* $z := h\mathrm{n} \in \mathrm{L}^2(\Gamma_\mathrm{N},\mathbb{R}^d)$*, and where* prox_ϕ *stands for the proximal operator associated with the Tresca friction functional* ϕ *defined by*

$$\begin{array}{rccl} \phi : & \mathrm{H}^1_\mathrm{D}(\Omega,\mathbb{R}^d) & \longrightarrow & \mathbb{R} \\ & w & \longmapsto & \phi(w) := \displaystyle\int_{\Gamma_\mathrm{N}} g\,\|w_\tau\|\,. \end{array}$$

Remark 3.15. The assumption that almost every point of Γ_N is in $\mathrm{int}_\Gamma(\Gamma_\mathrm{N})$ is only used to prove that a weak solution to the general Tresca friction problem (TP) is also a (strong) solution (precisely to get the Tresca friction law pointwisely on Γ_N). Of course, some sets do not satisfy this assumption, for instance the well-known Smith–Volterra–Cantor set (see, e.g, [5, Example 6.15 Section 6 Chapter 1]). Nevertheless it is trivially satisfied in most of standard cases found in practice. Furthermore, if this assumption is not satisfied, one can also prove that the weak solution to the general Tresca friction problem (TP) is a (strong) solution by adding the assumption that $g \in \mathrm{L}^\infty(\Gamma_\mathrm{N})$, and by using the isometry between the dual of $(\mathrm{L}^1(\Gamma_\mathrm{N},\mathbb{R}^d), \|\cdot\|_{\mathrm{L}^1(\Gamma_\mathrm{N},\mathbb{R}^d)_g})$ and $\mathrm{L}^\infty(\Gamma_\mathrm{N},\mathbb{R}^d)$ (with its standard norm $\|\cdot\|_{\mathrm{L}^\infty(\Gamma_\mathrm{N},\mathbb{R}^d)}$) where $\|\cdot\|_{\mathrm{L}^1(\Gamma_\mathrm{N},\mathbb{R}^d)_g}$ is the norm defined by

$$\begin{array}{rccl} \|\cdot\|_{\mathrm{L}^1(\Gamma_\mathrm{N},\mathbb{R}^d)_g} : & \mathrm{L}^1(\Gamma_\mathrm{N},\mathbb{R}^d) & \longrightarrow & \mathbb{R} \\ & w & \longmapsto & \displaystyle\int_{\Gamma_\mathrm{N}} g\,\|w\|\,. \end{array}$$

We refer to [15, Chapitre 3] for details in a similar context.

3.2 Sensitivity analysis of the Tresca friction problem

In this section we perform the sensitivity analysis of the Tresca friction problem. To this aim we consider the parameterized Tresca friction problem given by

$$\begin{cases} -\mathrm{div}(\mathrm{Ae}(u_t)) = f_t & \text{in } \Omega, \\ u_t = 0 & \text{on } \Gamma_{\mathrm{D}}, \\ \sigma_{\mathrm{n}}(u_t) = h_t & \text{on } \Gamma_{\mathrm{N}}, \\ \|\sigma_\tau(u_t)\| \leq g_t \text{ and } u_{t_\tau} \cdot \sigma_\tau(u_t) + g_t \|u_{t_\tau}\| = 0 & \text{on } \Gamma_{\mathrm{N}}, \end{cases} \tag{TP$_t$}$$

where $f_t \in \mathrm{L}^2(\Omega, \mathbb{R}^d)$, $h_t \in \mathrm{L}^2(\Gamma_{\mathrm{N}})$ and $g_t \in \mathrm{L}^2(\Gamma_{\mathrm{N}})$ such that $g_t > 0$ *a.e.* on Γ_{N}, for all $t \geq 0$.

3.2.1 Parameterized Tresca friction functional and twice epi-differentiability

Let us introduce the parameterized Tresca friction functional given by

$$\begin{array}{rccl} \Phi: & \mathbb{R}_+ \times \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d) & \longrightarrow & \mathbb{R} \\ & (t, w) & \longmapsto & \Phi(t, w) := \displaystyle\int_{\Gamma_{\mathrm{N}}} g_t \|w_\tau\| . \end{array} \tag{3.5}$$

From Proposition 3.14, the unique solution to the parameterized Tresca friction problem (TP$_t$) is given by

$$u_t = \mathrm{prox}_{\Phi(t,\cdot)}(F_t),$$

where F_t is the unique solution to the parameterized Dirichlet-Neumann problem

$$\begin{cases} -\mathrm{div}(\mathrm{Ae}(F_t)) = f_t & \text{in } \Omega, \\ F_t = 0 & \text{on } \Gamma_{\mathrm{D}}, \\ \mathrm{Ae}(F_t)\mathrm{n} = h_t \mathrm{n} & \text{on } \Gamma_{\mathrm{N}}, \end{cases} \tag{DN$_t$}$$

for all $t \geq 0$. Similarly to the scalar case (see [10]), since the parameterized Tresca friction functional depends on a parameter $t \geq 0$, we have to use the notion of twice epi-differentiability depending on a parameter (see Definition 2.12), in order to apply Theorem 2.13. Let us prepare the background for the twice epi-differentiability of the parameterized Tresca friction functional. More specifically, let us start with the characterization of the convex subdifferential of $\Phi(0, \cdot)$ (see Definition 2.2). To this aim, for all $s \in \Gamma_{\mathrm{N}}$, we introduce the *tangential norm map* defined by

$$\begin{array}{rccl} \left\|\cdot_{\tau(s)}\right\|: & \mathbb{R}^d & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \left\|x_{\tau(s)}\right\|, \end{array}$$

and we introduce an auxiliary problem defined, for all $u \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$, by

$$\begin{cases} -\mathrm{div}(\mathrm{Ae}(v)) = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \Gamma_{\mathrm{D}}, \\ \sigma_{\mathrm{n}}(v) = 0 & \text{on } \Gamma_{\mathrm{N}}, \\ \sigma_\tau(v)(s) \in g_0(s) \partial \left\|\cdot_{\tau(s)}\right\| (u(s)) & \text{on } \Gamma_{\mathrm{N}}, \end{cases} \tag{AP$_u$}$$

where, for almost all $s \in \Gamma_{\mathrm{N}}$, $\partial \left\|\cdot_{\tau(s)}\right\| (u(s))$ stands for the convex subdifferential of the tangential norm map $\left\|\cdot_{\tau(s)}\right\|$ at $u(s) \in \mathbb{R}^d$. For a given $u \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$, a solution to this problem (AP$_u$) is a function $v \in \mathrm{H}^1(\Omega, \mathbb{R}^d)$ such that $-\mathrm{div}(\mathrm{Ae}(v)) = 0$ in $\mathcal{D}'(\Omega, \mathbb{R}^d)$, $v = 0$ *a.e.* on Γ_{D}, $\mathrm{Ae}(v)\mathrm{n} \in \mathrm{L}^2(\Gamma_{\mathrm{N}}, \mathbb{R}^d)$ with $\sigma_{\mathrm{n}}(v) = 0$ *a.e.* on Γ_{N} and $\sigma_\tau(v)(s) \in g_0(s)\partial||\cdot_{\tau(s)}||(u(s))$ for almost all $s \in \Gamma_{\mathrm{N}}$.

Lemma 3.16. *Let* $u \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$. *Then*

$$\partial\Phi(0, \cdot)(u) = \textit{the set of solutions to Problem } (\mathrm{AP}_u).$$

Proof. Let $u \in \mathrm{H}^1_\mathrm{D}(\Omega,\mathbb{R}^d)$ and let us prove the two inclusions. Firstly, let $v \in \mathrm{H}^1(\Omega,\mathbb{R}^d)$ be a solution to Problem (AP_u). Then $v \in \mathrm{H}^1_\mathrm{D}(\Omega,\mathbb{R}^d)$, $\mathrm{Ae}(v)\mathrm{n} \in \mathrm{L}^2(\Gamma_\mathrm{N},\mathbb{R}^d)$ with $\sigma_\tau(v)(s) \in g_0(s)\partial||\cdot_{\tau(s)}||(u(s))$ for almost all $s \in \Gamma_\mathrm{N}$. Hence one has

$$\sigma_\tau(v)(s)\cdot(w_\tau(s)-u_\tau(s)) \leq g_0(s)(\|w_\tau(s)\|-\|u_\tau(s)\|),$$

for all $w \in \mathrm{H}^1_\mathrm{D}(\Omega,\mathbb{R}^d)$ and for almost all $s \in \Gamma_\mathrm{N}$. It follows that

$$\int_{\Gamma_\mathrm{N}} \sigma_\tau(v)\cdot(w_\tau-u_\tau) \leq \int_{\Gamma_\mathrm{N}} g_0\,\|w_\tau\| - \int_{\Gamma_\mathrm{N}} g_0\,\|u_\tau\|\,,$$

for all $w \in \mathrm{H}^1_\mathrm{D}(\Omega,\mathbb{R}^d)$. Moreover $-\mathrm{div}(\mathrm{Ae}(v)) = 0$ in $\mathcal{D}'(\Omega,\mathbb{R}^d)$, thus it holds $-\mathrm{div}(\mathrm{Ae}(v)) = 0$ in $\mathrm{L}^2(\Omega,\mathbb{R}^d)$. Hence, from divergence formula (see Proposition 2.16), one gets

$$\langle v, w-u\rangle_{\mathrm{H}^1_\mathrm{D}(\Omega,\mathbb{R}^d)} = \langle \mathrm{Ae}(v)\mathrm{n}, w-u\rangle_{\mathrm{H}^{-1/2}_{00}(\Gamma_\mathrm{N},\mathbb{R}^d)\times\mathrm{H}^{1/2}_{00}(\Gamma_\mathrm{N},\mathbb{R}^d)}\,,$$

for all $w \in \mathrm{H}^1_\mathrm{D}(\Omega,\mathbb{R}^d)$. Since $\mathrm{Ae}(v)\mathrm{n} \in \mathrm{L}^2(\Gamma_\mathrm{N},\mathbb{R}^d)$ and $\sigma_\mathrm{n}(v) = 0$ *a.e.* on Γ_N, one deduces that

$$\langle v, w-u\rangle_{\mathrm{H}^1_\mathrm{D}(\Omega,\mathbb{R}^d)} = \int_{\Gamma_\mathrm{N}} \sigma_\tau(v)\cdot(w_\tau-u_\tau)\,,$$

for all $w \in \mathrm{H}^1_\mathrm{D}(\Omega,\mathbb{R}^d)$. Therefore it follows that

$$\langle v, w-u\rangle_{\mathrm{H}^1_\mathrm{D}(\Omega,\mathbb{R}^d)} \leq \int_{\Gamma_\mathrm{N}} g_0\,\|w_\tau\| - \int_{\Gamma_\mathrm{N}} g_0\,\|u_\tau\|\,,$$

for all $w \in \mathrm{H}^1_\mathrm{D}(\Omega,\mathbb{R}^d)$. Thus $v \in \partial\Phi(0,\cdot)(u)$ and the first inclusion is proved. Conversely let $v \in \partial\Phi(0,\cdot)(u)$. Then one has

$$\langle v, w-u\rangle_{\mathrm{H}^1_\mathrm{D}(\Omega,\mathbb{R}^d)} \leq \int_{\Gamma_\mathrm{N}} g_0\,\|w_\tau\| - \int_{\Gamma_\mathrm{N}} g_0\,\|u_\tau\|\,, \tag{3.6}$$

for all $w \in \mathrm{H}^1_\mathrm{D}(\Omega,\mathbb{R}^d)$. Considering the function $w := u \pm \psi \in \mathrm{H}^1_\mathrm{D}(\Omega,\mathbb{R}^d)$ with any function $\psi \in \mathcal{D}(\Omega,\mathbb{R}^d)$, one deduces from Inequality (3.6) that $-\mathrm{div}(\mathrm{Ae}(v)) = 0$ in $\mathcal{D}'(\Omega,\mathbb{R}^d)$, thus it holds $-\mathrm{div}(\mathrm{Ae}(v)) = 0$ in $\mathrm{L}^2(\Omega,\mathbb{R}^d)$. Hence, from divergence formula (see Proposition 2.16) and Inequality (3.6), it follows that

$$\langle \mathrm{Ae}(v)\mathrm{n}, w-u\rangle_{\mathrm{H}^{-1/2}_{00}(\Gamma_\mathrm{N},\mathbb{R}^d)\times\mathrm{H}^{1/2}_{00}(\Gamma_\mathrm{N},\mathbb{R}^d)} \leq \int_{\Gamma_\mathrm{N}} g_0\,\|w_\tau\| - \int_{\Gamma_\mathrm{N}} g_0\,\|u_\tau\|\,,$$

for all $w \in \mathrm{H}^1_\mathrm{D}(\Omega,\mathbb{R}^d)$, and thus also for all $w \in \mathrm{H}^{1/2}_{00}(\Gamma_\mathrm{N},\mathbb{R}^d)$. Now, by considering $w := u+\varphi \in \mathrm{H}^{1/2}_{00}(\Gamma_\mathrm{N},\mathbb{R}^d)$, for any $\varphi \in \mathrm{H}^{1/2}_{00}(\Gamma_\mathrm{N},\mathbb{R}^d)$, one gets

$$\langle \mathrm{Ae}(v)\mathrm{n}, \varphi\rangle_{\mathrm{H}^{-1/2}_{00}(\Gamma_\mathrm{N},\mathbb{R}^d)\times\mathrm{H}^{1/2}_{00}(\Gamma_\mathrm{N},\mathbb{R}^d)} \leq \int_{\Gamma_\mathrm{N}} g_0\,\|\varphi_\tau\| \leq \|g_0\|_{\mathrm{L}^2(\Gamma_\mathrm{N})}\,\|\varphi\|_{\mathrm{L}^2(\Gamma_\mathrm{N},\mathbb{R}^d)}\,,$$

for all $\varphi \in \mathrm{H}^{1/2}_{00}(\Gamma_\mathrm{N},\mathbb{R}^d)$. From Proposition 2.15, one deduces that $\mathrm{Ae}(v)\mathrm{n} \in \mathrm{L}^2(\Gamma_\mathrm{N},\mathbb{R}^d)$ and also that

$$\begin{aligned}\int_{\Gamma_\mathrm{N}} \mathrm{Ae}(v)\mathrm{n}\cdot(w-u) &= \int_{\Gamma_\mathrm{N}} \sigma_\tau(v)\cdot(w_\tau-u_\tau) + \int_{\Gamma_\mathrm{N}} \sigma_\mathrm{n}(v)\,(w_\mathrm{n}-u_\mathrm{n})\\ &\leq \int_{\Gamma_\mathrm{N}} g_0\,\|w_\tau\| - \int_{\Gamma_\mathrm{N}} g_0\,\|u_\tau\|\,, \end{aligned} \tag{3.7}$$

for all $w \in \mathrm{H}^{1/2}_{00}(\Gamma_{\mathrm{N}},\mathbb{R}^d)$, and thus for all $w \in \mathrm{L}^2(\Gamma_{\mathrm{N}},\mathbb{R}^d)$ by density. By considering $w := u \pm \psi \mathrm{n} \in \mathrm{L}^2(\Gamma_{\mathrm{N}},\mathbb{R}^d)$ in Inequality (3.7), for any $\psi \in \mathrm{L}^2(\Gamma_{\mathrm{N}})$, one gets

$$\int_{\Gamma_{\mathrm{N}}} \sigma_{\mathrm{n}}(v)\psi = 0.$$

Therefore $\sigma_{\mathrm{n}}(v) = 0$ *a.e.* on Γ_{N} and Inequality (3.7) becomes

$$\int_{\Gamma_{\mathrm{N}}} \sigma_\tau(v) \cdot (w_\tau - u_\tau) \leq \int_{\Gamma_{\mathrm{N}}} g_0 \left\| w_\tau \right\| - \int_{\Gamma_{\mathrm{N}}} g_0 \left\| u_\tau \right\|, \tag{3.8}$$

for all $w \in \mathrm{L}^2(\Gamma_{\mathrm{N}},\mathbb{R}^d)$. Now let $s_0 \in \Gamma_{\mathrm{N}}$ be a Lebesgue point of $(\sigma_\tau(v))_i \in \mathrm{L}^2(\Gamma_{\mathrm{N}})$ for $i \in [[1,d]]$, $\sigma_\tau(v)\cdot u_\tau \in \mathrm{L}^1(\Gamma_{\mathrm{N}})$, $g_0 \in \mathrm{L}^2(\Gamma_{\mathrm{N}})$ and of $g_0 \left\| u_\tau \right\| \in \mathrm{L}^1(\Gamma_{\mathrm{N}})$, such that $s_0 \in \mathrm{int}_\Gamma(\Gamma_{\mathrm{N}})$. Let us consider the function $w \in \mathrm{L}^2(\Gamma_{\mathrm{N}},\mathbb{R}^d)$ defined by

$$w := \begin{cases} x & \text{on } B_\Gamma(s_0,\varepsilon), \\ u & \text{on } \Gamma_{\mathrm{N}} \backslash B_\Gamma(s_0,\varepsilon), \end{cases}$$

with $x \in \mathbb{R}^d$ and $\varepsilon > 0$ such that $B_\Gamma(s_0,\varepsilon) \subset \Gamma_{\mathrm{N}}$. Then one has from Inequality (3.8)

$$\frac{1}{|B_\Gamma(s_0,\varepsilon)|}\int_{B_\Gamma(s_0,\varepsilon)} \sigma_\tau(v)\cdot(x_\tau - u_\tau) \leq \frac{1}{|B_\Gamma(s_0,\varepsilon)|}\int_{B_\Gamma(s_0,\varepsilon)} g_0 \left\| x_\tau \right\| - \frac{1}{|B_\Gamma(s_0,\varepsilon)|}\int_{B_\Gamma(s_0,\varepsilon)} g_0 \left\| u_\tau \right\|.$$

The map $s \in \Gamma \mapsto \left\| x_{\tau(s)} \right\| \in \mathbb{R}_+$ is continuous since $\mathrm{n} \in \mathcal{C}^0(\Gamma)$, thus s_0 is a Lebesgue point of $g_0 \left\| x_\tau \right\| \in \mathrm{L}^2(\Gamma_{\mathrm{N}})$, then $\sigma_\tau(v)(s_0)\cdot(x_{\tau(s_0)} - u_\tau(s_0)) \leq g_0(s_0)\left\| x_{\tau(s_0)} \right\| - g_0(s_0)\left\| u_\tau(s_0) \right\|$ by letting $\varepsilon \to 0^+$. This inequality is true for any $x \in \mathbb{R}^d$, therefore $\sigma_\tau(v)(s_0) \in g_0(s_0)\partial||\cdot_{\tau(s_0)}||(u(s_0))$. Moreover, almost every point of Γ_{N} are in $\mathrm{int}_\Gamma(\Gamma_{\mathrm{N}})$ and are Lesbegue points of $(\sigma_\tau(v))_i \in \mathrm{L}^2(\Gamma_{\mathrm{N}})$ for $i \in [[1,d]]$, $\sigma_\tau(v)\cdot u_\tau \in \mathrm{L}^1(\Gamma_{\mathrm{N}})$, $g_0 \in \mathrm{L}^2(\Gamma_{\mathrm{N}})$ and of $g_0 \left\| u_\tau \right\| \in \mathrm{L}^1(\Gamma_{\mathrm{N}})$, hence one deduces

$$\sigma_\tau(v)(s) \in g_0(s)\partial||\cdot_{\tau(s)}||(u(s)),$$

for almost all $s \in \Gamma_{\mathrm{N}}$, and this proves the second inclusion. □

Remark 3.17. As one can see in the proof of Lemma 3.16, the assumption that Γ is of class $\mathcal{C}^1$ is only used to ensure that $\mathrm{n} \in \mathcal{C}^0(\Gamma)$, and thus to characterize the convex subdifferential of $\Phi(0,\cdot)$ as the set of solutions to Problem (AP_u).

Since the twice epi-differentiability is defined using the second-order difference quotient functions, let us compute the second-order difference quotient functions of Φ at $u \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ for $v \in \partial\Phi(0,\cdot)(u)$.

Proposition 3.18. *For all $t > 0$, all $u \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ and all $v \in \partial\Phi(0,\cdot)(u)$, it holds that*

$$\Delta_t^2\Phi(u \mid v)(w) = \int_{\Gamma_{\mathrm{N}}} \Delta_t^2 G(s)(u(s) \mid \sigma_\tau(v)(s))(w(s))\,\mathrm{d}s, \tag{3.9}$$

for all $w \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$, where, for almost all $s \in \Gamma_{\mathrm{N}}$, $\Delta_t^2 G(s)(u(s) \mid \sigma_\tau(v)(s))$ stands for the second-order difference quotient function of $G(s)$ at $u(s) \in \mathbb{R}^d$ for $\sigma_\tau(v)(s) \in g_0(s)\partial||\cdot_{\tau(s)}||(u(s))$, with $G(s)$ defined by

$$\begin{array}{rrcl} G(s): & \mathbb{R}_+ \times \mathbb{R}^d & \longrightarrow & \mathbb{R} \\ & (t,x) & \longmapsto & G(s)(t,x) := g_t(s)\left\| x_{\tau(s)} \right\|. \end{array}$$

Remark 3.19. Note that, for almost all $s \in \Gamma_{\mathrm{N}}$ and all $t \geq 0$, $G(s)(t,\cdot) := g_t(s)\left\|\cdot_{\tau(s)}\right\|$ is a proper lower semi-continuous convex function on $\mathbb{R}^d$. Moreover, since $g_0 > 0$ *a.e.* on Γ_{N}, it follows that

$$\partial\left[G(s)(0,\cdot)\right](x) = g_0(s)\partial\|\cdot_{\tau(s)}\|(x),$$

for all $x \in \mathbb{R}^d$ and for almost all $s \in \Gamma_{\mathrm{N}}$.

Proof of Proposition 3.18. Let $t > 0$, $u \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ and $v \in \partial\Phi(0,\cdot)(u)$. From Lemma 3.16 and the divergence formula (see Proposition 2.16), one deduces that

$$\langle v, w\rangle_{\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)} = \int_{\Gamma_{\mathrm{N}}} \sigma_\tau(v)\cdot w,$$

for all $w \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$. It follows that

$$\Delta^2_t\Phi(u \mid v)(w) = \int_{\Gamma_{\mathrm{N}}} \frac{g_t(s)\left\|u_\tau(s) + t w_\tau(s)\right\| - g_t(s)\left\|u_\tau(s)\right\| - t\sigma_\tau(v)(s)\cdot w(s)}{t^2}\,\mathrm{d}s,$$

for all $w \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$. Moreover, since $\sigma_\tau(v)(s) \in g_0(s)\partial\|\cdot_{\tau(s)}\|(u(s))$ for almost all $s \in \Gamma_{\mathrm{N}}$, one deduces that

$$\Delta^2_t\Phi(u \mid v)(w) = \int_{\Gamma_{\mathrm{N}}} \Delta^2_t G(s)(u(s) \mid \sigma_\tau(v)(s))(w(s))\,\mathrm{d}s,$$

for all $w \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$, which concludes the proof. □

From Proposition 3.18, it is clear that the twice epi-differentiability of the parameterized Tresca friction functional Φ is related to the twice epi-differentiability of the parameterized function $G(s)$. Hence we have to compute the second-order epi-derivative of $G(s)$ for almost all $s \in \Gamma_{\mathrm{N}}$. To this aim, let us start with the computation of the twice epi-differentiability of the tangential norm map.

Lemma 3.20. *For all $s \in \Gamma_{\mathrm{N}}$, the map $\left\|\cdot_{\tau(s)}\right\|$ is twice epi-differentiable at any $x \in \mathbb{R}^d$ for any $y \in \partial\left\|\cdot_{\tau(s)}\right\|(x)$ and its second-order epi-derivative is given by*

$$\mathrm{d}^2_e\left\|\cdot_{\tau(s)}\right\|(x \mid y)(z) = \begin{cases} \frac{1}{2\left\|x_{\tau(s)}\right\|}\left(\left\|z_{\tau(s)}\right\|^2 - \left|z_{\tau(s)}\cdot\frac{x_{\tau(s)}}{\left\|x_{\tau(s)}\right\|}\right|^2\right) & \text{if } x_{\tau(s)} \neq 0,\\ \iota_{\mathrm{N}_{\overline{\mathrm{B}(0,1)}\cap(\mathbb{R}\mathrm{n}(s))^\perp}(y)}(z) & \text{if } x_{\tau(s)} = 0,\end{cases}$$

for all $z \in \mathbb{R}^d$, where $\mathrm{N}_{\overline{\mathrm{B}(0,1)}\cap(\mathbb{R}\mathrm{n}(s))^\perp}(y)$ is the normal cone to $\overline{\mathrm{B}(0,1)}\cap(\mathbb{R}\mathrm{n(s)})^\perp$ at y.

Proof. Let $s \in \Gamma_{\mathrm{N}}$. Note that

$$\partial\left\|\cdot_{\tau(s)}\right\|(x) := \begin{cases} \left\{\frac{x_{\tau(s)}}{\left\|x_{\tau(s)}\right\|}\right\} & \text{if } x_{\tau(s)} \neq 0,\\ \overline{\mathrm{B}(0,1)}\cap(\mathbb{R}\mathrm{n(s)})^\perp & \text{if } x_{\tau(s)} = 0,\end{cases}$$

and that $\left\|\cdot_{\tau(s)}\right\| = \xi_{\overline{\mathrm{B}(0,1)}\cap(\mathbb{R}\mathrm{n(s)})^\perp}$, where $\xi_{\overline{\mathrm{B}(0,1)}\cap(\mathbb{R}\mathrm{n(s)})^\perp}$ is the support function of $\overline{\mathrm{B}(0,1)}\cap(\mathbb{R}\mathrm{n(s)})^\perp$ which is a nonempty convex closed subset of $\mathbb{R}^d$. Moreover, since

$$(\overline{\mathrm{B}(0,1)}\cap(\mathbb{R}\mathrm{n(s)})^\perp)^\perp = \mathbb{R}\mathrm{n}(s),$$

one can apply Proposition 2.11 to get that

$$\mathrm{d}^2_e\left\|\cdot_{\tau(s)}\right\|(x \mid y) = \iota_{\mathrm{N}_{\overline{\mathrm{B}(0,1)}\cap(\mathbb{R}\mathrm{n}(s))^\perp}(y)},$$

for all $x \in \mathbb{R}\mathrm{n}(s)$ and all $y \in \overline{\mathrm{B}(0,1)} \cap (\mathbb{R}\mathrm{n}(s))^{\perp}$. In the case where $x \notin \mathbb{R}\mathrm{n}(s)$ (i.e. $x_{\tau(s)} \neq 0$), one can easily prove that $\left\|\cdot_{\tau(s)}\right\|$ is twice Fréchet differentiable at x with

$$\mathrm{D}^2 \left\|\cdot_{\tau(s)}\right\|(x)(z_1,z_2) = \frac{1}{\left\|x_{\tau(s)}\right\|}\left(z_{1_{\tau(s)}} \cdot z_{2_{\tau(s)}} - \left(x_{\tau(s)} \cdot z_{2_{\tau(s)}}\right) z_{1_{\tau(s)}} \cdot \frac{x_{\tau(s)}}{\left\|x_{\tau(s)}\right\|^2} \right), \qquad \forall\, (z_1,z_2) \in \mathbb{R}^d \times \mathbb{R}^d.$$

From Remark 2.9, one gets

$$\mathrm{d}_e^2 \left\|\cdot_{\tau(s)}\right\| \left(x \mid \frac{x_{\tau(s)}}{\left\|x_{\tau(s)}\right\|} \right)(z) = \frac{1}{2}\mathrm{D}^2 \left\|\cdot_{\tau(s)}\right\|(x)(z,z) = \frac{1}{2\left\|x_{\tau(s)}\right\|}\left(\left\|z_{\tau(s)}\right\|^2 - \left| z_{\tau(s)} \cdot \frac{x_{\tau(s)}}{\left\|x_{\tau(s)}\right\|} \right|^2 \right),$$

for all $z \in \mathbb{R}^d$, which concludes the proof. □

Now, with additional assumptions, let us compute the second-order epi-derivative of $G(s)$ for almost all $s \in \Gamma_{\mathrm{N}}$.

Proposition 3.21. *Assume that, for almost all $s \in \Gamma_{\mathrm{N}}$, the map $t \in \mathbb{R}_+ \mapsto g_t(s) \in \mathbb{R}_+$ is differentiable at $t = 0$, with its derivative denoted by $g_0'(s)$. Then, for almost all $s \in \Gamma_{\mathrm{N}}$, the map $G(s)$ is twice epi-differentiable at any $x \in \mathbb{R}^d$ for all $y \in g_0(s)\partial||\cdot_{\tau(s)}||(x)$ with*

$$\mathrm{D}_e^2 G(s)(x \mid y)(z) := \begin{cases} \frac{g_0(s)}{2\left\|x_{\tau(s)}\right\|}\left(\left\|z_{\tau(s)}\right\|^2 - \left| z_{\tau(s)} \cdot \frac{x_{\tau(s)}}{\left\|x_{\tau(s)}\right\|} \right|^2 \right) + g_0'(s)\frac{x_{\tau(s)}}{\left\|x_{\tau(s)}\right\|} \cdot z & \text{if } x_{\tau(s)} \neq 0, \\ \iota_{\mathrm{N}_{\overline{\mathrm{B}(0,1)} \cap (\mathbb{R}\mathrm{n}(s))^{\perp}}\left(\frac{y}{g_0(s)}\right)}(z) + g_0'(s)\frac{y}{g_0(s)} \cdot z & \text{if } x_{\tau(s)} = 0, \end{cases}$$

for all $z \in \mathbb{R}^d$.

Proof. We use the same notations as in Definitions 2.8 and 2.12. Let $x \in \mathbb{R}^d$. Then, for almost all $s \in \Gamma_{\mathrm{N}}$, for all $y \in g_0(s)\partial||\cdot_{\tau(s)}||(x)$ and all $z \in \mathbb{R}^d$, one has

$$\begin{aligned} \Delta_t^2 G(s)(x \mid y)(z) &= \frac{g_t(s)\left\|x_{\tau(s)} + tz_{\tau(s)}\right\| - g_t(s)\left\|x_{\tau(s)}\right\| - ty \cdot z}{t^2} \\ &= g_t(s)\frac{\left\|x_{\tau(s)} + tz_{\tau(s)}\right\| - \left\|x_{\tau(s)}\right\| - t\frac{y}{g_0(s)} \cdot z}{t^2} + \frac{(g_t(s) - g_0(s))}{tg_0(s)}y \cdot z, \end{aligned}$$

that is

$$\Delta_t^2 G(s)(x \mid y)(z) = g_t(s)\delta_t^2||\cdot_{\tau(s)}||\left(x \mid \frac{y}{g_0(s)} \right)(z) + \frac{(g_t(s) - g_0(s))}{tg_0(s)}y \cdot z,$$

with $\frac{y}{g_0(s)} \in \partial||\cdot_{\tau(s)}||(x)$, and where $\delta_t^2||\cdot_{\tau(s)}||(x|\frac{y}{g_0(s)})$ is the second-order difference quotient function of $||\cdot_{\tau(s)}||$ at x for $\frac{y}{g_0(s)}$ (see Definition 2.8 since $||\cdot_{\tau(s)}||$ is a t-independent function). Using the characterization of Mosco epi-convergence (see Proposition 2.7) and Lemma 3.20, one gets

$$\mathrm{D}_e^2 G(s)(x \mid y)(z) = g_0(s)\mathrm{d}_e^2 \left\|\cdot_{\tau(s)}\right\|\left(x \mid \frac{y}{g_0(s)} \right) + g_0'(s)\frac{y}{g_0(s)} \cdot z.$$

The proof is complete. □

To conclude this part, let us characterize $\mathrm{N}_{\overline{\mathrm{B}(0,1)} \cap (\mathbb{R}\mathrm{n}(s))^{\perp}}(y)$ for almost all $s \in \Gamma_{\mathrm{N}}$.

Lemma 3.22. *Let $s \in \Gamma_{\mathrm{N}}$. It holds that*

$$\mathrm{N}_{\overline{\mathrm{B}(0,1)}\cap(\mathbb{R}\mathrm{n}(s))^{\perp}}(y) = \begin{cases} \mathbb{R}\mathrm{n}(s) & \text{if } y \in \mathrm{B}(0,1) \cap (\mathbb{R}\mathrm{n}(s))^{\perp}, \\ \mathbb{R}\mathrm{n}(s) + \mathbb{R}_+ y & \text{if } y \in \partial\mathrm{B}(0,1) \cap (\mathbb{R}\mathrm{n}(s))^{\perp}, \end{cases}$$

for all $y \in \overline{\mathrm{B}(0,1)} \cap (\mathbb{R}\mathrm{n}(s))^{\perp}$, where $\mathbb{R}_+ y := \{z \in \mathbb{R}^d \mid \exists \nu \geq 0$ such that $z = \nu y\}$.

Proof. Let $s \in \Gamma_{\mathrm{N}}$ and $y \in \overline{\mathrm{B}(0,1)} \cap (\mathbb{R}\mathrm{n}(s))^{\perp}$.

(i) First, let $y \in \mathrm{B}(0,1) \cap (\mathbb{R}\mathrm{n}(s))^{\perp}$. If $v \in \mathbb{R}\mathrm{n}(s)$, then

$$v \cdot (y - z) = 0, \qquad \forall z \in \overline{\mathrm{B}(0,1)} \cap (\mathbb{R}\mathrm{n}(s))^{\perp},$$

thus $v \in \mathrm{N}_{\overline{\mathrm{B}(0,1)}\cap(\mathbb{R}\mathrm{n}(s))^{\perp}}(y)$. Since this is true for any $v \in \mathbb{R}\mathrm{n}(s)$, one deduces that

$$\mathbb{R}\mathrm{n}(s) \subset \mathrm{N}_{\overline{\mathrm{B}(0,1)}\cap(\mathbb{R}\mathrm{n}(s))^{\perp}}(y).$$

Consider $v \in \mathrm{N}_{\overline{\mathrm{B}(0,1)}\cap(\mathbb{R}\mathrm{n}(s))^{\perp}}(y)$. Then it holds that

$$v \cdot (z - y) \leq 0, \qquad \forall z \in \overline{\mathrm{B}(0,1)} \cap (\mathbb{R}\mathrm{n}(s))^{\perp}.$$

Moreover there exists $\varepsilon > 0$ such that $\mathrm{B}(y,\varepsilon) \cap (\mathbb{R}\mathrm{n}(s))^{\perp} \subset \mathrm{B}(0,1) \cap (\mathbb{R}\mathrm{n}(s))^{\perp}$. Therefore by considering $z := y + \varepsilon \frac{w}{2\|w\|}$ for any $w \in (\mathbb{R}\mathrm{n}(s))^{\perp}$, one deduces that

$$v \cdot w = 0, \qquad \forall w \in (\mathbb{R}\mathrm{n}(s))^{\perp}.$$

Thus $v \in ((\mathbb{R}\mathrm{n}(s))^{\perp})^{\perp} = \mathbb{R}\mathrm{n}(s)$. Since this is true for any $v \in \mathrm{N}_{\overline{\mathrm{B}(0,1)}\cap(\mathbb{R}\mathrm{n}(s))^{\perp}}(y)$, one deduces that

$$\mathrm{N}_{\overline{\mathrm{B}(0,1)}\cap(\mathbb{R}\mathrm{n}(s))^{\perp}}(y) \subset \mathbb{R}\mathrm{n}(s).$$

(ii) Let $y \in \partial\mathrm{B}(0,1) \cap (\mathbb{R}\mathrm{n}(s))^{\perp}$. If $v \in \mathbb{R}\mathrm{n}(s) + \mathbb{R}_+ y$, then

$$v \cdot (z - y) = v_{\tau(s)} \cdot (z - y) \leq \left\|v_{\tau(s)}\right\| \|z\| - v_{\tau(s)} \cdot y \leq \left\|v_{\tau(s)}\right\| - \left\|v_{\tau(s)}\right\| = 0,$$

for all $z \in \overline{\mathrm{B}(0,1)} \cap (\mathbb{R}\mathrm{n}(s))^{\perp}$. Thus it follows that

$$\mathbb{R}\mathrm{n}(s) + \mathbb{R}_+ y \subset \mathrm{N}_{\overline{\mathrm{B}(0,1)}\cap(\mathbb{R}\mathrm{n}(s))^{\perp}}(y).$$

Let $v \in \mathrm{N}_{\overline{\mathrm{B}(0,1)}\cap(\mathbb{R}\mathrm{n}(s))^{\perp}}(y)$, and consider $z := \frac{1}{2}\left(\frac{v_{\tau(s)}}{\left\|v_{\tau(s)}\right\|}\|y\| + y\right) \in \overline{\mathrm{B}(0,1)} \cap (\mathbb{R}\mathrm{n}(s))^{\perp}$. One deduces that

$$0 \geq v \cdot (z - y) = v_{\tau(s)} \cdot \frac{1}{2}\left(\frac{v_{\tau(s)}}{\left\|v_{\tau(s)}\right\|}\|y\| - y\right) = \frac{1}{2}\left(\left\|v_{\tau(s)}\right\| \|y\| - v_{\tau(s)} \cdot y\right) \geq 0,$$

thus $\left\|v_{\tau(s)}\right\| \|y\| = v_{\tau(s)} \cdot y$, hence $v_{\tau(s)} \in \mathbb{R}_+ y$. Since this is true for any $v \in \mathrm{N}_{\overline{\mathrm{B}(0,1)}\cap(\mathbb{R}\mathrm{n}(s))^{\perp}}(y)$, one deduces that

$$\mathrm{N}_{\overline{\mathrm{B}(0,1)}\cap(\mathbb{R}\mathrm{n}(s))^{\perp}}(y) \subset \mathbb{R}\mathrm{n}(s) + \mathbb{R}_+ y.$$

The proof is complete. □

3.2.2 The derivative of the solution to the parameterized Tresca friction problem

From the previous results and some additional assumptions detailed below, we are now in a position to state and prove the main result of this paper which characterizes the derivative of the solution to the parameterized Tresca friction problem (TP_t).

Theorem 3.23. *Let $u_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ be the unique solution to the parameterized Tresca friction problem (TP_t) for all $t \geq 0$. Let us assume that:*

(i) *the map $t \in \mathbb{R}_+ \mapsto f_t \in \mathrm{L}^2(\Omega,\mathbb{R}^d)$ is differentiable at $t = 0$, with its derivative denoted by $f'_0 \in \mathrm{L}^2(\Omega,\mathbb{R}^d)$;*

(ii) *the map $t \in \mathbb{R}_+ \mapsto h_t \in \mathrm{L}^2(\Gamma_{\mathrm{N}})$ is differentiable at $t = 0$, with its derivative denoted by $h'_0 \in \mathrm{L}^2(\Gamma_{\mathrm{N}})$;*

(iii) *for almost all $s \in \Gamma_{\mathrm{N}}$, the map $t \in \mathbb{R}_+ \mapsto g_t(s) \in \mathbb{R}_+$ is differentiable at $t = 0$, with its derivative denoted by $g'_0(s)$, and also $g'_0 \in \mathrm{L}^2(\Gamma_{\mathrm{N}})$;*

(iv) *the map $s \in \Gamma_{\mathrm{N}_{\mathrm{R}}^{u_0,g_0}} \mapsto \frac{g_0(s)}{\|u_{0_\tau}(s)\|} \in \mathbb{R}_+$ belongs to $\mathrm{L}^4(\Gamma_{\mathrm{N}_{\mathrm{R}}^{u_0,g_0}})$ (see below for the set $\Gamma_{\mathrm{N}_{\mathrm{R}}^{u_0,g_0}}$);*

(v) *the parameterized Tresca friction functional Φ defined in (3.5) is twice epi-differentiable (see Definition 2.12) at u_0 for $F_0 - u_0 \in \partial\Phi(0,\cdot)(u_0)$, with*

$$\mathrm{D}^2_e\Phi(u_0 \mid F_0 - u_0)(w) = \int_{\Gamma_{\mathrm{N}}} \mathrm{D}^2_e G(s)(u_0(s) \mid \sigma_\tau(F_0 - u_0)(s))(w(s))\,\mathrm{d}s, \tag{3.10}$$

for all $w \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$, where $F_0 \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ is the unique solution to the parameterized Dirichlet-Neumann problem (DN_t) for the parameter $t = 0$.

Then the map $t \in \mathbb{R}_+ \mapsto u_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ is differentiable at $t = 0$, and its derivative denoted by $u'_0 \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ is the unique weak solution to the tangential Signorini problem

$$\left\{\begin{array}{rcll}
-\mathrm{div}(\mathrm{A}\mathrm{e}(u'_0)) & = & f'_0 & \text{in } \Omega,\\
u'_0 & = & 0 & \text{on } \Gamma_{\mathrm{D}},\\
\sigma_{\mathrm{n}}(u'_0) & = & h'_0 & \text{on } \Gamma_{\mathrm{N}},\\
u'_{0_\tau} & = & 0 & \text{on } \Gamma_{\mathrm{N}_{\mathrm{T}}^{u_0,g_0}},\\
\sigma_\tau(u'_0) + \frac{g_0}{\|u_{0_\tau}\|}\left(u'_{0_\tau} - \left(u'_{0_\tau}\cdot\frac{u_{0_\tau}}{\|u_{0_\tau}\|}\right)\frac{u_{0_\tau}}{\|u_{0_\tau}\|}\right) & = & -g'_0\frac{u_{0_\tau}}{\|u_{0_\tau}\|} & \text{on } \Gamma_{\mathrm{N}_{\mathrm{R}}^{u_0,g_0}},\\
u'_{0_\tau} \in \mathbb{R}_-\frac{\sigma_\tau(u_0)}{g_0}, \left(\sigma_\tau(u'_0) - g'_0\frac{\sigma_\tau(u_0)}{g_0}\right)\cdot\frac{\sigma_\tau(u_0)}{g_0} \leq 0 & & & \\
\text{and } u'_{0_\tau}\cdot\left(\sigma_\tau(u'_0) - g'_0\frac{\sigma_\tau(u_0)}{g_0}\right) & = & 0 & \text{on } \Gamma_{\mathrm{N}_{\mathrm{S}}^{u_0,g_0}},
\end{array}\right. \tag{SP'_0}$$

where Γ_{N} is decomposed (up to a null set) as $\Gamma_{\mathrm{N}_{\mathrm{T}}^{u_0,g_0}} \cup \Gamma_{\mathrm{N}_{\mathrm{R}}^{u_0,g_0}} \cup \Gamma_{\mathrm{N}_{\mathrm{S}}^{u_0,g_0}}$ with

$$\begin{aligned}
\Gamma_{\mathrm{N}_{\mathrm{R}}^{u_0,g_0}} &:= \left\{s \in \Gamma_{\mathrm{N}} \mid u_{0_\tau}(s) \neq 0\right\},\\
\Gamma_{\mathrm{N}_{\mathrm{T}}^{u_0,g_0}} &:= \left\{s \in \Gamma_{\mathrm{N}} \mid u_{0_\tau}(s) = 0 \text{ and } \frac{\sigma_\tau(u_0)(s)}{g_0(s)} \in \mathrm{B}(0,1)\cap(\mathbb{R}\mathrm{n}(s))^\perp\right\},\\
\Gamma_{\mathrm{N}_{\mathrm{S}}^{u_0,g_0}} &:= \left\{s \in \Gamma_{\mathrm{N}} \mid u_{0_\tau}(s) = 0 \text{ and } \frac{\sigma_\tau(u_0)(s)}{g_0(s)} \in \partial\mathrm{B}(0,1)\cap(\mathbb{R}\mathrm{n}(s))^\perp\right\}.
\end{aligned}$$

Remark 3.24. *As mentioned in our previous papers [3, 10], one can naturally expect from Proposition 3.18 that the second-order epi-derivative of the parameterized Tresca friction functional Φ*

at u_0 for $F_0 - u_0$ is given by Equality (3.10)*, which corresponds to the inversion of the symbols* ME-lim *and* $\int_{\Gamma_{\mathrm{N}}}$ *in Equality* (3.9)*. Nevertheless, to the best of our knowledge, the validity of this inversion is an open question in the literature. Precisely, we do not know, in general, if the parameterized Tresca friction functional is twice epi-differentiable at u_0 for $F_0 - u_0$. Nevertheless, similarly to [10, Appendix A], one can prove it in some practical situations.*

Proof of Theorem 3.23. From Hypotheses (iii), (v) and Proposition 3.21, it follows that

$$\mathrm{D}_e^2\Phi(u_0 \mid F_0 - u_0)(w) = \int_{\Gamma_{\mathrm{N}_{\mathrm{R}}^{u_0,g_0}}} \left(\frac{g_0}{2\left\|u_{0_\tau}\right\|}\left(\|w_\tau\|^2 - \left|w_\tau \cdot \frac{u_{0_\tau}}{\|u_{0_\tau}\|}\right|^2\right) + g_0' \frac{u_{0_\tau}}{\|u_{0_\tau}\|}\cdot w\right)$$
$$+ \int_{\Gamma_{\mathrm{N}}\backslash\Gamma_{\mathrm{N}_{\mathrm{R}}^{u_0,g_0}}} \iota_{\mathrm{N}_{\overline{\mathrm{B}(0,1)}\cap(\mathbb{R}\mathrm{n}(s))^\perp}\left(\frac{\sigma_\tau(F_0-u_0)(s)}{g_0(s)}\right)}(w(s))\mathrm{d}s + \int_{\Gamma_{\mathrm{N}}\backslash\Gamma_{\mathrm{N}_{\mathrm{R}}^{u_0,g_0}}} g_0' \frac{\sigma_\tau\left(F_0 - u_0\right)}{g_0}\cdot w,$$

which can be rewritten as

$$\mathrm{D}_e^2\Phi(u_0 \mid F_0 - u_0)(w) =$$
$$\Psi(w) + \int_{\Gamma_{\mathrm{N}_{\mathrm{R}}^{u_0,g_0}}} g_0' \frac{u_{0_\tau}}{\|u_{0_\tau}\|}\cdot w_\tau + \iota_{\mathcal{K}_{u_0,\frac{\sigma_\tau(F_0-u_0)}{g_0}}}(w) + \int_{\Gamma_{\mathrm{N}}\backslash\Gamma_{\mathrm{N}_{\mathrm{R}}^{u_0,g_0}}} g_0' \frac{\sigma_\tau\left(F_0 - u_0\right)}{g_0}\cdot w_\tau,$$

for all $w \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$, where Ψ is defined by

$$\begin{array}{rccl}\Psi: & \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d) & \longrightarrow & \mathbb{R} \\ & w & \longmapsto & \Psi(w) := \displaystyle\int_{\Gamma_{\mathrm{N}_{\mathrm{R}}^{u_0,g_0}}} \frac{g_0}{2\left\|u_{0_\tau}\right\|}\left(\|w_\tau\|^2 - \left|w_\tau \cdot \frac{u_{0_\tau}}{\|u_{0_\tau}\|}\right|^2\right),\end{array}$$

which is well defined from the continuous embedding $\mathrm{H}^1(\Omega,\mathbb{R}^d) \hookrightarrow \mathrm{L}^4(\Gamma,\mathbb{R}^d)$ and from Hypothesis (iv), and where $\mathcal{K}_{u_0,\frac{\sigma_\tau(F_0-u_0)}{g_0}}$ is the nonempty closed convex subset of $\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ defined by

$$\mathcal{K}_{u_0,\frac{\sigma_\tau(F_0-u_0)}{g_0}} := \Bigg\{ w \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d) \mid w(s) \in \mathrm{N}_{\overline{\mathrm{B}(0,1)}\cap(\mathbb{R}\mathrm{n}(s))^\perp}\left(\frac{\sigma_\tau\left(F_0 - u_0\right)(s)}{g_0(s)}\right) \text{ for almost all } s \in \Gamma_{\mathrm{N}}\backslash\Gamma_{\mathrm{N}_{\mathrm{R}}^{u_0,g_0}} \Bigg\}.$$

Moreover, from Lemma 3.22 and since $\sigma_\tau(F_0) = 0$ *a.e.* on Γ_{N}, it follows that

$$\mathcal{K}_{u_0,\frac{\sigma_\tau(F_0-u_0)}{g_0}} = \left\{ w \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d) \mid w_\tau = 0 \text{ a.e. on } \Gamma_{\mathrm{N}_{\mathrm{T}}^{u_0,g_0}} \text{ and } w_\tau \in \mathbb{R}_- \frac{\sigma_\tau(u_0)}{g_0} \text{ a.e. on } \Gamma_{\mathrm{N}_{\mathrm{S}}^{u_0,g_0}} \right\}.$$

Since $\frac{g_0}{\|u_{0_\tau}\|} > 0$ *a.e.* on $\Gamma_{\mathrm{N}_{\mathrm{R}}^{u_0,g_0}}$ and from Lemma 3.9, one deduces that Ψ is convex and Fréchet differentiable on $\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$. In particular we get that $\mathrm{D}_e^2\Phi(u_0|F_0 - u_0)$ is a proper lower semi-continuous convex function on $\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$. Moreover, from Hypotheses (i) and (ii) and from the linearity of the Dirichlet-Neumann problem (DN) and Proposition 3.4, we can easily prove that the map $t \in \mathbb{R}_+ \mapsto F_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ is differentiable at $t = 0$, with its derivative $F_0' \in \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ being the unique solution to the Dirichlet-Neumann problem

$$\begin{cases} -\mathrm{div}(\mathrm{Ae}(F_0')) = f_0' & \text{in } \Omega, \\ F_0' = 0 & \text{on } \Gamma_{\mathrm{D}}, \\ \mathrm{Ae}(F_0')\mathrm{n} = h_0'\mathrm{n} & \text{on } \Gamma_{\mathrm{N}}. \end{cases}$$

Thus one can apply Theorem 2.13 to deduce that the map $t \in \mathbb{R}_+ \mapsto u_t \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ is differentiable at $t = 0$, and its derivative $u_0' \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ satisfies

$$u_0' = \mathrm{prox}_{\mathrm{D}^2_e\Phi(u_0|F_0-u_0)}(F_0'),$$

which, from the definition of the proximal operator (see Proposition 2.3), leads to

$$F_0' - u_0' \in \partial \mathrm{D}^2_e\Phi(u_0 \mid F_0 - u_0)(u_0'),$$

which means that

$$\langle F_0' - u_0', w - u_0' \rangle_{\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)} \leq \mathrm{D}^2_e\Phi(u_0 \mid F_0 - u_0)(w) - \mathrm{D}^2_e\Phi(u_0 \mid F_0 - u_0)(u_0'),$$

for all $w \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$. Hence we get that

$$\begin{aligned}\langle F_0' - u_0', w - u_0' \rangle_{\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)} \leq{}& \Psi(w) - \Psi(u_0') + \iota_{\mathcal{K}_{u_0, \frac{\sigma_\tau(F_0-u_0)}{g_0}}}(w) - \iota_{\mathcal{K}_{u_0, \frac{\sigma_\tau(F_0-u_0)}{g_0}}}(u_0') \\ &+ \int_{\Gamma_{\mathrm{N}^{u_0,g_0}_{\mathrm{R}}}} g_0' \frac{u_{0_\tau}}{\|u_{0_\tau}\|} \cdot \left(w_\tau - u'_{0_\tau}\right) + \int_{\Gamma_{\mathrm{N}} \backslash \Gamma_{\mathrm{N}^{u_0,g_0}_{\mathrm{R}}}} g_0' \frac{\sigma_\tau\left(F_0 - u_0\right)}{g_0} \cdot \left(w_\tau - u'_{0_\tau}\right),\end{aligned}$$

for all $w \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$. Moreover, since $\sigma_\tau(F_0) = 0$ *a.e.* on Γ_{N}, and for all $w \in \mathcal{K}_{u_0, \frac{\sigma_\tau(F_0-u_0)}{g_0}}$, $w_\tau = 0$ *a.e.* $\Gamma_{\mathrm{N}^{u_0,g_0}_{\mathrm{T}}}$, one deduces that $u_0' \in \mathcal{K}_{u_0, \frac{\sigma_\tau(F_0-u_0)}{g_0}}$ and

$$\begin{aligned}\langle u_0', w - u_0' \rangle_{\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)} + \Psi(w) - \Psi(u_0') \geq{}& \int_\Omega f_0' \cdot (w - u_0') + \int_{\Gamma_{\mathrm{N}}} h_0' \left(w_{\mathrm{n}} - u'_{0_{\mathrm{n}}}\right) \\ &- \int_{\Gamma_{\mathrm{N}^{u_0,g_0}_{\mathrm{R}}}} g_0' \frac{u_{0_\tau}}{\|u_{0_\tau}\|} \cdot \left(w_\tau - u'_{0_\tau}\right) + \int_{\Gamma_{\mathrm{N}^{u_0,g_0}_{\mathrm{S}}}} g_0' \frac{\sigma_\tau\left(u_0\right)}{g_0} \cdot \left(w_\tau - u'_{0_\tau}\right),\end{aligned}$$

for all $w \in \mathcal{K}_{u_0, \frac{\sigma_\tau(F_0-u_0)}{g_0}}$. Moreover, since Ψ is convex and Fréchet differentiable on $\mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ (see Lemma 3.9), one gets from Proposition 2.4 that

$$\begin{aligned}\langle \nabla\Psi(u_0'), w - u_0' \rangle_{\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)} \geq{}& -\langle u_0', w - u_0' \rangle_{\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)} + \int_\Omega f_0' \cdot (w - u_0') + \int_{\Gamma_{\mathrm{N}}} h_0' \left(w_{\mathrm{n}} - u'_{0_{\mathrm{n}}}\right) \\ &- \int_{\Gamma_{\mathrm{N}^{u_0,g_0}_{\mathrm{R}}}} g_0' \frac{u_{0_\tau}}{\|u_{0_\tau}\|} \cdot \left(w_\tau - u'_{0_\tau}\right) + \int_{\Gamma_{\mathrm{N}^{u_0,g_0}_{\mathrm{S}}}} g_0' \frac{\sigma_\tau\left(u_0\right)}{g_0} \cdot \left(w_\tau - u'_{0_\tau}\right),\end{aligned}$$

for all $w \in \mathcal{K}_{u_0, \frac{\sigma_\tau(F_0-u_0)}{g_0}}$. Finally, using the expression of $\nabla\Psi(u_0') \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$, one gets

$$\begin{aligned}\langle u_0', w - u_0' \rangle_{\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)} \geq{}& \int_\Omega f_0' \cdot (w - u_0') + \int_{\Gamma_{\mathrm{N}}} h_0' \left(w_{\mathrm{n}} - u'_{0_{\mathrm{n}}}\right) + \int_{\Gamma_{\mathrm{N}^{u_0,g_0}_{\mathrm{S}}}} g_0' \frac{\sigma_\tau\left(u_0\right)}{g_0} \cdot \left(w_\tau - u'_{0_\tau}\right) \\ &+ \int_{\Gamma_{\mathrm{N}^{u_0,g_0}_{\mathrm{R}}}} \left(-g_0' \frac{u_{0_\tau}}{\|u_{0_\tau}\|} - \frac{g_0}{\|u_{0_\tau}\|}\left(u'_{0_\tau} - \left(u'_{0_\tau} \cdot \frac{u_{0_\tau}}{\|u_{0_\tau}\|}\right)\frac{u_{0_\tau}}{\|u_{0_\tau}\|}\right)\right) \cdot \left(w_\tau - u'_{0_\tau}\right),\end{aligned}$$

for all $w \in \mathcal{K}_{u_0, \frac{\sigma_\tau(F_0-u_0)}{g_0}}$. From Definition 3.6 one deduces that u_0' is the unique weak solution to the tangential Signorini problem (SP_0') which concludes the proof. □

Remark 3.25. Consider the framework of Theorem 3.23. Note that u_0' is the unique weak solution to the tangential Signorini problem (SP_0'), but is not necessarily a strong solution. Nevertheless, in the case where $\mathrm{Ae}(u_0')\mathrm{n} \in \mathrm{L}^2(\Gamma_{\mathrm{N}}, \mathbb{R}^d)$ and the decomposition $\Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{N}^{u_0,g_0}_{\mathrm{T}}} \cup \Gamma_{\mathrm{N}^{u_0,g_0}_{\mathrm{R}}} \cup \Gamma_{\mathrm{N}^{u_0,g_0}_{\mathrm{S}}}$ of Γ is consistent (see Definition 3.7), then u_0' is a strong solution to the tangential Signorini problem (SP_0').

4 Application to optimal control

Consider the functional framework introduced at the beginning of Section 3. Let $f \in \mathrm{L}^2(\Omega, \mathbb{R}^d)$, $h \in \mathrm{L}^2(\Gamma_\mathrm{N})$, $g_1 \in \mathrm{L}^\infty(\Gamma_\mathrm{N})$ such that $g_1 \geq m$ *a.e.* on Γ_N for some positive constant $m > 0$ and $g_2 \in \mathrm{L}^\infty(\Gamma_\mathrm{N})$ such that $||g_2||_{\mathrm{L}^\infty(\Gamma_\mathrm{N})} > 0$. In this section we consider the optimal control problem given by

$$\underset{z \in \mathcal{U}}{\text{minimize}} \; \mathcal{J}(z), \tag{4.1}$$

where $\mathcal{J}$ is the cost functional defined by

$$\begin{array}{rccl} \mathcal{J}: & \mathrm{V} & \longrightarrow & \mathbb{R} \\ & z & \longmapsto & \mathcal{J}(z) := \frac{1}{2} \left\| u(\ell(z)) \right\|^2_{\mathrm{H}^1_\mathrm{D}(\Omega, \mathbb{R}^d)} + \frac{\beta}{2} \left\| \ell(z) \right\|^2_{\mathrm{L}^2(\Gamma_\mathrm{N})}, \end{array} \tag{4.2}$$

where V is the open subset of $\mathrm{L}^\infty(\Gamma_\mathrm{N})$ defined by

$$\mathrm{V} := \left\{ z \in \mathrm{L}^\infty(\Gamma_\mathrm{N}) \mid \exists C(z) > 0, \, \ell(z) > C(z) \text{ } a.e. \text{ on } \Gamma_\mathrm{N} \right\},$$

where ℓ is the map defined by $z \in \mathrm{L}^\infty(\Gamma_\mathrm{N}) \mapsto \ell(z) := g_1 + z g_2 \in \mathrm{L}^\infty(\Gamma_\mathrm{N})$, and where $u(\ell(z)) \in \mathrm{H}^1_\mathrm{D}(\Omega, \mathbb{R}^d)$ stands for the unique solution to the Tresca friction problem given by

$$\left\{ \begin{aligned} -\mathrm{div}(\mathrm{Ae}(u)) &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma_\mathrm{D}, \\ \sigma_\mathrm{n}(u) &= h && \text{on } \Gamma_\mathrm{N}, \\ \|\sigma_\tau(u)\| \leq \ell(z) \text{ and } u_\tau \cdot \sigma_\tau(u) + \ell(z) \|u_\tau\| &= 0 && \text{on } \Gamma_\mathrm{N}, \end{aligned} \right. \tag{CTP$_{\ell(z)}$}$$

where $\beta > 0$ is a positive constant and where $\mathcal{U}$ is a given nonempty convex subset of V such that $\mathcal{U}$ is a bounded closed subset of $\mathrm{L}^2(\Gamma_\mathrm{N})$. Note that the first term in the cost functional $\mathcal{J}$ corresponds to the compliance, while the second term is the energy consumption which is standard in optimal control problems (see, e.g., [21]).

This section is organized as follows. In Subsection 4.1 we prove the existence of a solution to Problem (4.1). In Subsection 4.2 we prove, under some assumptions, that $\mathcal{J}$ is Gateaux differentiable on V and we characterize its gradient. Finally, in Subsection 4.3, numerical simulations are performed to solve Problem (4.1) on a two-dimensional example.

4.1 Existence of a solution

This section is dedicated to the following existence result.

Proposition 4.1. *There exists $z^* \in \mathcal{U}$ such that $\mathcal{J}(z^*) \leq \mathcal{J}(z)$ for all $z \in \mathcal{U}$.*

Proof. In this proof the strong (resp. weak) convergence in Hilbert spaces is denoted by $\rightarrow$ (resp. $\rightharpoonup$) and all limits with respect to the index i will be considered for $i \rightarrow +\infty$. Since $0 \leq \mathcal{J}(z) < +\infty$ for all $z \in \mathcal{U}$, we get that $\inf_{z \in \mathcal{U}} \mathcal{J}(z) \in \mathbb{R}_+$. Considering a minimizing sequence $(z_i)_{i \in \mathbb{N}}$, there exists $N \in \mathbb{N}$ such that $\mathcal{J}(z_i) \leq 1 + \inf_{z \in \mathcal{U}} \mathcal{J}(z)$ for all $i \geq N$, that is

$$\frac{1}{2} \left\| u(g_i) \right\|^2_{\mathrm{H}^1_\mathrm{D}(\Omega, \mathbb{R}^d)} + \frac{\beta}{2} \left\| g_i \right\|^2_{\mathrm{L}^2(\Gamma_\mathrm{N})} \leq 1 + \inf_{z \in \mathcal{U}} \mathcal{J}(z),$$

for all $i \geq N$, where $g_i := g_1 + z_i g_2$. Thus the sequence $(g_i)_{i \in \mathbb{N}}$ is bounded in $\mathrm{L}^2(\Gamma_\mathrm{N})$ and thus, up to a subsequence that we do not relabel, weakly converges to some $g^* \in \mathrm{L}^2(\Gamma_\mathrm{N})$. Moreover, since $\mathcal{U}$ is a bounded closed convex subset of $\mathrm{L}^2(\Gamma_\mathrm{N})$ (and thus weakly closed in $\mathrm{L}^2(\Gamma_\mathrm{N})$), we know that, up to a subsequence that we do not relabel, the sequence $(z_i)_{i\in\mathbb{N}}$ weakly converges to some $z^* \in \mathcal{U}$. Moreover one has

$$\left| \int_{\Gamma_{\mathrm{N}}} (g_i - g_1 - z^* g_2)\, w \right| = \int_{\Gamma_{\mathrm{N}}} (z_i - z^*)\, g_2 w,$$

for all $w \in \mathrm{L}^2(\Gamma_{\mathrm{N}})$, and, since $g_2 \in \mathrm{L}^\infty(\Gamma_{\mathrm{N}})$, it holds that $g_2 w \in \mathrm{L}^2(\Gamma_{\mathrm{N}})$ and one deduces that $g_i \rightharpoonup g_1 + z^* g_2$ in $\mathrm{L}^2(\Gamma_{\mathrm{N}})$ and thus $g^* = g_1 + z^* g_2$. In a similar way, up to a subsequence that we do not relabel, the sequence $(u(g_i))_{i\in\mathbb{N}}$ weakly converges in $\mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ to some $u^* \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ and thus $u(g_i) \to u^*$ in $\mathrm{L}^2(\Gamma, \mathbb{R}^d)$ from the compact embedding $\mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d) \hookrightarrow\!\!\!\!\rightarrow \mathrm{L}^2(\Gamma, \mathbb{R}^d)$ (see Proposition 2.14). Let us prove that $u(g_i) \to u^*$ in $\mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$. It holds that

$$\|u^* - u(g_i)\|^2_{\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)} = \langle u^*, u^* - u(g_i)\rangle_{\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)} - \langle u(g_i), u^* - u(g_i)\rangle_{\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)},$$

for all $i \in \mathbb{N}$. Using the weak formulation satisfied by $u(g_i)$, we get that

$$\begin{aligned}
\|u^* - u(g_i)\|^2_{\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)} &\leq \langle u^*, u^* - u(g_i)\rangle_{\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)} - \int_\Omega f \cdot (u^* - u(g_i)) \\
&\qquad - \int_{\Gamma_{\mathrm{N}}} h\,(u^*_{\mathrm{n}} - u(g_i)_{\mathrm{n}}) + \int_{\Gamma_{\mathrm{N}}} g_i\,(\|u^*_\tau\| - \|u(g_i)_\tau\|) \\
&\leq \langle u^*, u^* - u(g_i)\rangle_{\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)} - \int_\Omega f \cdot (u^* - u(g_i)) - \int_{\Gamma_{\mathrm{N}}} h\,(u^*_{\mathrm{n}} - u(g_i)_{\mathrm{n}}) \\
&\qquad + C\,\|u^* - u(g_i)\|_{\mathrm{L}^2(\Gamma,\mathbb{R}^d)} \longrightarrow 0,
\end{aligned}$$

where $C \geq 0$ is a constant (depending only on Ω and on $\max_{i\in\mathbb{N}} ||g_i||_{\mathrm{L}^2(\Gamma_{\mathrm{N}})}$). Now let us prove that $u^* = u(g_1 + z^* g_2)$. For $w \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ fixed, it holds that

$$\langle u(g_i), w - u(g_i)\rangle_{\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)} + \int_{\Gamma_{\mathrm{N}}} g_i\,\|w_\tau\| - \int_{\Gamma_{\mathrm{N}}} g_i\,\|u(g_i)_\tau\| \geq \int_\Omega f \cdot (w - u(g_i)), \tag{4.3}$$

for all $i \in \mathbb{N}$. Note that:

(i) $\left|\langle u(g_i), w - u(g_i)\rangle_{\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)} - \langle u^*, w - u^*\rangle_{\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)}\right| \leq D\,\|u^* - u(g_i)\|_{\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)} \longrightarrow 0;$

(ii) $\left|\int_\Omega f \cdot (w - u(g_i)) - \int_\Omega f \cdot (w - u^*)\right| \leq D\,\|f\|_{\mathrm{L}^2(\Omega,\mathbb{R}^d)}\,\|u^* - u(g_i)\|_{\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)} \longrightarrow 0;$

(iii) $\left|\int_{\Gamma_{\mathrm{N}}} g_i\,(\|w_\tau\| - \|u(g_i)_\tau\|) - \int_{\Gamma_{\mathrm{N}}} g^*\,(\|w_\tau\| - \|u^*_\tau\|)\right| \leq$
$\left|\int_{\Gamma_{\mathrm{N}}} (g_i - g^*)\,\|w_\tau\|\right| + \left|\int_{\Gamma_{\mathrm{N}}} (g_i - g^*)\,\|u^*_\tau\|\right| + D\,\|u^* - u(g_i)\|_{\mathrm{L}^2(\Gamma,\mathbb{R}^d)} \longrightarrow 0;$

where $D \geq 0$ is a constant (depending only on Ω, A and w). Therefore it follows in (4.3) when $i \to +\infty$ that

$$\langle u^*, w - u^*\rangle_{\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)} + \int_{\Gamma_{\mathrm{N}}} g^*\,\|w_\tau\| - \int_{\Gamma_{\mathrm{N}}} g^*\,\|u^*_\tau\| \geq \int_\Omega f \cdot (w - u^*).$$

Since this inequality is true for all $w \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$ and $g^* = g_1 + z^* g_2$, one deduces that $u^* = u(g_1 + z^* g_2)$, and then

$$\begin{aligned}
\mathcal{J}(z^*) &= \frac{1}{2}\,\|u(g_1 + z^* g_2)\|^2_{\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)} + \frac{\beta}{2}\,\|g_1 + z^* g_2\|^2_{\mathrm{L}^2(\Gamma_{\mathrm{N}})} \leq \\
&\qquad \liminf_{i\to+\infty} \left(\frac{1}{2}\,\|u(g_i)\|^2_{\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)} + \frac{\beta}{2}\,\|g_i\|^2_{\mathrm{L}^2(\Gamma_{\mathrm{N}})}\right) \leq \liminf_{i\to+\infty} \mathcal{J}(z_k) = \inf_{z\in\mathcal{U}} \mathcal{J}(z),
\end{aligned}$$

which concludes the proof. □

Remark 4.2. Since the solution to the Tresca friction problem is not linear with respect to the friction term, note that $\mathcal{J}$ is not a strictly convex functional (and thus the uniqueness of the solution to Problem (4.1) is not guaranteed).

4.2 Gateaux differentiability of the cost functional

Consider the auxiliary functional

$$\begin{array}{rccl} \mathrm{J}: & \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)\times\mathrm{L}^\infty(\Gamma_{\mathrm{N}}) & \longrightarrow & \mathbb{R} \\ & (v,g) & \longmapsto & \mathrm{J}(v,g) := \frac{1}{2}\|v\|^2_{\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)} + \frac{\beta}{2}\|g\|^2_{\mathrm{L}^2(\Gamma_{\mathrm{N}})}. \end{array} \tag{4.4}$$

One can easily prove that J is Fréchet differentiable on $\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)\times\mathrm{L}^\infty(\Gamma_{\mathrm{N}})$ and its Fréchet differential at some $(v,g)\in\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)\times\mathrm{L}^\infty(\Gamma_{\mathrm{N}})$, denoted by $\mathrm{dJ}(v,g)$, is given by

$$\mathrm{dJ}(v,g)(v_1,g_1) = \langle v,v_1\rangle_{\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)} + \beta\,\langle g,g_1\rangle_{\mathrm{L}^2(\Gamma_{\mathrm{N}})}\,,$$

for all $(v_1,g_1)\in\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)\times\mathrm{L}^\infty(\Gamma_{\mathrm{N}})$. Now let us introduce the map

$$\begin{array}{rccl} \mathcal{F}: & \mathrm{V} & \longrightarrow & \mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)\times\mathrm{L}^\infty(\Gamma_{\mathrm{N}}) \\ & z & \longmapsto & \mathcal{F}(z) := (u(\ell(z)),\ell(z))\,, \end{array} \tag{4.5}$$

where $u(\ell(z))\in\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ is the unique solution to the Tresca friction problem ($\mathrm{CTP}_{\ell(z)}$). Hence the cost functional $\mathcal{J}$ is given by the composition $\mathcal{J} = J\circ\mathcal{F}$.

Theorem 4.3. *Let $z_0\in\mathrm{V}$ be fixed and let us denote by $u_0 := u(\ell(z_0))$. Assume that:*

(i) *the map $s\in\Gamma_{\mathrm{N}^{u_0,\ell(z_0)}_{\mathrm{R}}}\mapsto\frac{\ell(z_0)(s)}{\|u_{0_\tau}(s)\|}\in\mathbb{R}^*_+$ belongs to $\mathrm{L}^4(\Gamma_{\mathrm{N}^{u_0,\ell(z_0)}_{\mathrm{R}}})$ (see below for the set $\Gamma_{\mathrm{N}^{u_0,\ell(z_0)}_{\mathrm{R}}}$);*

(ii) *the parameterized Tresca friction functional Φ defined in (3.5) is twice epi-differentiable at u_0 for $F-u_0\in\partial\Phi(0,\cdot)(u_0)$, with*

$$\mathrm{D}^2_e\Phi(u_0)|F-u_0)(w) = \int_{\Gamma_{\mathrm{N}}}\mathrm{D}^2_eG(s)(u_0(s)|\sigma_\tau(F-u_0)(s))(w(s))\,\mathrm{d}s, \qquad \forall w\in\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d),$$

where, for almost all $s\in\Gamma_{\mathrm{N}}$, the map $G(s)$ is defined in Proposition 3.18, and $F\in\mathrm{H}^1_{\mathrm{D}}(\Omega,\mathbb{R}^d)$ is the unique solution to the Dirichlet-Neumann problem

$$\begin{cases} -\mathrm{div}(\mathrm{Ae}(F)) = f & \text{in } \Omega, \\ F = 0 & \text{on } \Gamma_{\mathrm{D}}, \\ \mathrm{Ae}(F)\mathrm{n} = h\mathrm{n} & \text{on } \Gamma_{\mathrm{N}}. \end{cases} \tag{4.6}$$

Then the cost functional $\mathcal{J}$ is Gateaux differentiable at z_0 and its differential $\mathrm{d}_G\mathcal{J}(z_0)$ is given by

$$\mathrm{d}_G\mathcal{J}(z_0)(z) = \int_{\Gamma_{\mathrm{N}^{u_0,\ell(z_0)}_{\mathrm{R}}}} zg_2\left(\beta\left(g_1+z_0g_2\right)-\|u_{0_\tau}\|\right) + \int_{\Gamma_{\mathrm{N}^{u_0,\ell(z_0)}_{\mathrm{T}}}\cup\Gamma_{\mathrm{N}^{u_0,\ell(z_0)}_{\mathrm{S}}}} \beta zg_2\left(g_1+z_0g_2\right),$$

for all $z\in\mathrm{L}^\infty(\Gamma_{\mathrm{N}})$, where Γ_{N} is decomposed (up to a null set) as $\Gamma_{\mathrm{N}^{u_0,\ell(z_0)}_{\mathrm{T}}}\cup\Gamma_{\mathrm{N}^{u_0,\ell(z_0)}_{\mathrm{R}}}\cup\Gamma_{\mathrm{N}^{u_0,\ell(z_0)}_{\mathrm{S}}}$ with

$$\begin{aligned} \Gamma_{\mathrm{N}^{u_0,\ell(z_0)}_{\mathrm{R}}} &:= \left\{s\in\Gamma_{\mathrm{N}}\mid u_{0_\tau}(s)\neq 0\right\}, \\ \Gamma_{\mathrm{N}^{u_0,\ell(z_0)}_{\mathrm{T}}} &:= \left\{s\in\Gamma_{\mathrm{N}}\mid u_{0_\tau}(s)=0 \text{ and } \tfrac{\sigma_\tau(u_0)(s)}{\ell(z_0)(s)}\in\mathrm{B}(0,1)\cap(\mathbb{R}\mathrm{n}(s))^\perp\right\}, \\ \Gamma_{\mathrm{N}^{u_0,\ell(z_0)}_{\mathrm{S}}} &:= \left\{s\in\Gamma_{\mathrm{N}}\mid u_{0_\tau}(s)=0 \text{ and } \tfrac{\sigma_\tau(u_0)(s)}{\ell(z_0)(s)}\in\partial\mathrm{B}(0,1)\cap(\mathbb{R}\mathrm{n}(s))^\perp\right\}. \end{aligned}$$

Proof. Let $z \in \mathrm{L}^\infty(\Gamma_\mathrm{N})$ and $t>0$ be sufficiently small such that $z_t := z_0 + tz \in \mathrm{V}$. We denote by $u_t := u(\ell(z_t)) \in \mathrm{H}^1_\mathrm{D}(\Omega,\mathbb{R}^d)$ and by $g_t := \ell(z_t) \in \mathrm{L}^\infty(\Gamma_\mathrm{N})$. From Subsection 3, $u_t \in \mathrm{H}^1_\mathrm{D}(\Omega,\mathbb{R}^d)$ is given by $u_t = \mathrm{prox}_{\Phi(t,\cdot)}(F)$, where Φ is the parameterized Tresca friction functional defined in (3.5) and F is the unique solution to the Dirichlet-Neumann problem (4.6). From Hypotheses (i), (ii) and since the map $t \in \mathbb{R}_+ \mapsto g_t \in \mathrm{L}^\infty(\Gamma_\mathrm{N})$ is differentiable at $t=0$, with its derivative given by $g_0' := zg_2$, one can apply Theorem 3.23 to deduce that the map $t \in \mathbb{R}_+ \mapsto u_t \in \mathrm{H}^1_\mathrm{D}(\Omega,\mathbb{R}^d)$ is differentiable at $t=0$ and its derivative, denoted by $u_0' \in \mathcal{K}_{u_0,\frac{\sigma_\tau(F_0-u_0)}{g_0}} \subset \mathrm{H}^1_\mathrm{D}(\Omega,\mathbb{R}^d)$, is the unique solution to the variational inequality (which is the weak formulation of a tangential Signorini problem) given by

$$\begin{aligned}\langle u_0', w-u_0'\rangle_{\mathrm{H}^1_\mathrm{D}(\Omega,\mathbb{R}^d)} \geq & \int_{\Gamma_{\mathrm{N}^{u_0,g_0}_\mathrm{S}}} g_0' \frac{\sigma_\tau(u_0)}{g_0}\cdot\left(w_\tau - u_{0_\tau}'\right) \\ & + \int_{\Gamma_{\mathrm{N}^{u_0,g_0}_\mathrm{R}}} \left(-g_0'\frac{u_{0_\tau}}{\|u_{0_\tau}\|} - \frac{g_0}{\|u_{0_\tau}\|}\left(u_{0_\tau}' - \left(u_{0_\tau}'\cdot\frac{u_{0_\tau}}{\|u_{0_\tau}\|}\right)\frac{u_{0_\tau}}{\|u_{0_\tau}\|}\right)\right)\cdot\left(w_\tau - u_{0_\tau}'\right),\end{aligned}$$

for all $w \in \mathcal{K}_{u_0,\frac{\sigma_\tau(F_0-u_0)}{g_0}}$, where

$$\mathcal{K}_{u_0,\frac{\sigma_\tau(F_0-u_0)}{g_0}} := \left\{ w \in \mathrm{H}^1_\mathrm{D}(\Omega,\mathbb{R}^d) \mid w_\tau = 0 \ a.e. \text{ on } \Gamma_{\mathrm{N}^{u_0,g_0}_\mathrm{T}} \text{ and } w_\tau \in \mathbb{R}_-\frac{\sigma_\tau(u_0)}{g_0} \ a.e. \text{ on } \Gamma_{\mathrm{N}^{u_0,g_0}_\mathrm{S}} \right\}.$$

Since $\mathcal{J} = \mathrm{J}\circ\mathcal{F}$, with J Fréchet differentiable on $\mathrm{H}^1_\mathrm{D}(\Omega,\mathbb{R}^d)\times\mathrm{V}$, and

$$\frac{\|\mathcal{F}(z_0+tz)-\mathcal{F}(z_0)-t\,(u_0',g_0')\|_{\mathrm{H}^1_\mathrm{D}(\Omega,\mathbb{R}^d)\times\mathrm{L}^\infty(\Gamma_\mathrm{N})}}{t} = \frac{\|u_t-u_0-tu_0'\|_{\mathrm{H}^1_\mathrm{D}(\Omega,\mathbb{R}^d)}}{t} \longrightarrow 0,$$

when $t\to 0^+$, we deduce that $\mathcal{J}$ has a right derivative at z_0 in the direction z given by

$$\mathcal{J}'(z_0)(z) = \langle u_0', u_0\rangle_{\mathrm{H}^1_\mathrm{D}(\Omega,\mathbb{R}^d)} + \beta\,\langle g_0, g_0'\rangle_{\mathrm{L}^2(\Gamma_\mathrm{N})}.$$

Furthermore, since $u_0' \pm u_0 \in \mathcal{K}_{u_0,\frac{\sigma_\tau(F-u_0)}{g_0}}$, one deduces that

$$\begin{aligned}&\langle u_0', u_0\rangle_{\mathrm{H}^1_\mathrm{D}(\Omega,\mathbb{R}^d)} = \\ &\qquad \int_{\Gamma_{\mathrm{N}^{u_0,g_0}_\mathrm{R}}} \left(-g_0'\frac{u_{0_\tau}}{\|u_{0_\tau}\|} - \frac{g_0}{\|u_{0_\tau}\|}\left(u_0'(g_0,g_0')_\tau - \left(u_0'(g_0,g_0')_\tau\cdot\frac{u_{0_\tau}}{\|u_{0_\tau}\|}\right)\frac{u_{0_\tau}}{\|u_{0_\tau}\|}\right)\right)\cdot u_{0_\tau}.\end{aligned}$$

Since

$$\int_{\Gamma_{\mathrm{N}^{u_0,g_0}_\mathrm{R}}} \frac{g_0}{\|u_{0_\tau}\|}\left(u_0'(g_0,g_0')_\tau - \left(u_0'(g_0,g_0')_\tau\cdot\frac{u_{0_\tau}}{\|u_{0_\tau}\|}\right)\frac{u_{0_\tau}}{\|u_{0_\tau}\|}\right)\cdot u_{0_\tau} = 0,$$

we get that

$$\langle u_0', u_0\rangle_{\mathrm{H}^1_\mathrm{D}(\Omega,\mathbb{R}^d)} = -\int_{\Gamma_{\mathrm{N}^{u_0,g_0}_\mathrm{R}}} g_0'\,\|u_{0_\tau}\|,$$

and we can rewrite the right derivative of $\mathcal{J}$ at z_0 in the direction z as

$$\mathcal{J}'(z_0)(z) = -\int_{\Gamma_{\mathrm{N}^{u_0,g_0}_\mathrm{R}}} g_0'\,\|u_{0_\tau}\| + \int_{\Gamma_\mathrm{N}} \beta g_0' g_0 = \int_{\Gamma_{\mathrm{N}^{u_0,g_0}_\mathrm{R}}} g_0'\,(\beta g_0 - \|u_{0_\tau}\|) + \int_{\Gamma_{\mathrm{N}^{u_0,g_0}_\mathrm{T}}\cup\Gamma_{\mathrm{N}^{u_0,g_0}_\mathrm{S}}} \beta g_0' g_0,$$

and thus

$$\mathcal{J}'(z_0)(z) = \int_{\Gamma_{\mathrm{N}_{\mathrm{R}}^{u_0,\ell(z_0)}}} zg_2\left(\beta\left(g_1+z_0g_2\right)-\|u_{0_\tau}\|\right) + \int_{\Gamma_{\mathrm{N}_{\mathrm{T}}^{u_0,\ell(z_0)}}\cup\Gamma_{\mathrm{N}_{\mathrm{S}}^{u_0,\ell(z_0)}}} \beta z g_2\left(g_1+z_0g_2\right).$$

Note that $\mathcal{J}'(z_0)$ is linear and continuous on $\mathrm{L}^\infty(\Gamma_\mathrm{N})$. Thus $\mathcal{J}$ is Gateaux differentiable at z_0 with its Gateaux differential given by $\mathrm{d}_G\mathcal{J}(z_0) := \mathcal{J}'(z_0)$. The proof is complete. □

Remark 4.4. In the proof of Theorem 4.3, note that the derivative u_0' depends on the pair $(g_0, g_0') = (g_1 + z_0g_2, zg_2)$ and thus on the term $z \in \mathrm{L}^\infty(\Gamma_\mathrm{N})$. Therefore let us denote by $u_0' := u_0'(z)$. Note that $u_0'(z)$ is not linear with respect to z. However one can observe that the scalar product $\langle u_0'(z), u_0\rangle_{\mathrm{H}^1_\mathrm{D}(\Omega,\mathbb{R}^d)}$, that appears in the proof of Theorem 4.3, is linear with respect to z. Therefore it leads to an expression of $\mathcal{J}'(z_0)$ that is linear with respect to z, and thus to the Gateaux differentiability of $\mathcal{J}$ at z_0.

4.3 Numerical simulations

In this section we assume that $||g_2||_{\mathrm{L}^\infty(\Gamma_\mathrm{N})} < m$, where $m > 0$ is the constant introduced at the beginning of Section 4 and we take the admissible set $\mathcal{U}$ given by

$$\mathcal{U} := \left\{ z \in \mathrm{L}^2(\Gamma_\mathrm{N}) \mid -1 \leq z \leq 1 \text{ } a.e. \text{ on } \Gamma_\mathrm{N} \right\},$$

which is a nonempty convex subset of V and is a bounded closed subset of $\mathrm{L}^2(\Gamma_\mathrm{N})$. In this section our aim is to numerically solve an example of Problem (4.1) in the two-dimensional case $d = 2$, by making use of our theoretical result obtained in Theorem 4.3.

4.3.1 Numerical methodology

Starting with an initial control $z_0 \in \mathcal{U}$, we compute $z_d \in \mathrm{L}^\infty(\Gamma_\mathrm{N})$ given by

$$z_d := \begin{cases} -g_2\left(\beta\left(g_1+z_0g_2\right)-\|u_{0\tau}\|\right) & \text{on } \Gamma_{\mathrm{N}_{\mathrm{R}}^{u_0,\ell(z_0)}}, \\ -\beta g_2\left(g_1+z_0g_2\right) & \text{on } \Gamma_{\mathrm{N}_{\mathrm{T}}^{u_0,\ell(z_0)}} \cup \Gamma_{\mathrm{N}_{\mathrm{S}}^{u_0,\ell(z_0)}}, \end{cases} \tag{4.7}$$

which is, from Theorem 4.3, a descent direction of the functional $\mathcal{J}$ at z_0 since it satisfies

$$\begin{aligned} \mathrm{d}_G\mathcal{J}(z_0)(z_d) = -||g_2\left(\beta\left(g_1+z_0g_2\right)-\|u_{0\tau}\|\right)||^2_{\mathrm{L}^2(\Gamma_{\mathrm{N}_{\mathrm{R}}^{u_0,\ell(z_0)}})} \\ - ||\beta g_2\left(g_1+z_0g_2\right)||^2_{\mathrm{L}^2(\Gamma_{\mathrm{N}_{\mathrm{T}}^{u_0,\ell(z_0)}}\cup\Gamma_{\mathrm{N}_{\mathrm{S}}^{u_0,\ell(z_0)}})} \leq 0. \end{aligned}$$

Then the control is updated as $z_1 = \mathrm{proj}_\mathcal{U}\left(z_0 + \eta z_d\right)$, where $\eta > 0$ is a fixed parameter and $\mathrm{proj}_\mathcal{U}$ is the classical projection operator onto $\mathcal{U}$ considered in $\mathrm{L}^2(\Gamma_\mathrm{N})$. Then the algorithm restarts with z_1, and so on.

Let us mention that the numerical simulations have been performed using Freefem++ software [16] with P1-finite elements and standard affine mesh. The Tresca friction problem is numerically solved using an adaptation of iterative switching algorithms (this adaptation is close to the one described in [3, Appendix C] which concerns a scalar Tresca friction problem). We also precise that, for all $i \in \mathbb{N}^*$, the difference between the cost functional $\mathcal{J}$ at the iteration $20 \times i$ and at the iteration $20 \times (i-1)$ is computed. The smallness of this difference is used as a stopping criterion for the algorithm.

4.3.2 Example and numerical results

In this subsection take $d = 2$ and let Ω be the unit disk of $\mathbb{R}^2$ with its boundary $\Gamma := \partial\Omega$ decomposed as $\Gamma = \Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{N}}$ (see Figure 1), where

$$\Gamma_{\mathrm{D}} := \left\{ (\cos\theta, \sin\theta) \in \Gamma \mid 0 \leq \theta \leq \tfrac{\pi}{2} \right\},$$
$$\Gamma_{\mathrm{N}} := \left\{ (\cos\theta, \sin\theta) \in \Gamma \mid \tfrac{\pi}{2} < \theta < 2\pi \right\}.$$

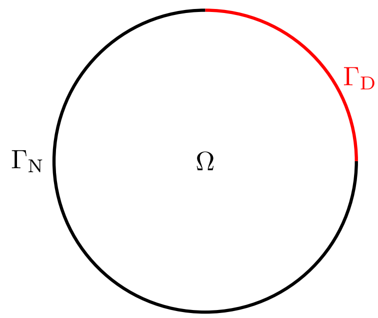

Figure 1: Unit disk Ω and its boundary $\Gamma = \Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{N}}$.

We assume that Ω is *isotropic*, in the sense that the Cauchy stress tensor is given by

$$\sigma(w) = 2\mu \mathrm{e}(w) + \lambda \mathrm{tr}\,(\mathrm{e}(w))\,\mathrm{I},$$

for all $w \in \mathrm{H}^1_{\mathrm{D}}(\Omega, \mathbb{R}^d)$, where $\mathrm{tr}\,(\mathrm{e}(w))$ is the trace of the matrix $\mathrm{e}(w)$ and where $\mu \geq 0$ and $\lambda \geq 0$ are Lamé parameters (see, e.g., [28]). In what follows we take $\mu = 0.3846$ and $\lambda = 0.5769$. This corresponds to a Young's modulus equal to 1 and to a Poisson's ratio equal to 0.3, which is a typical value for a large variety of materials. Let us consider the arbitrary functions $h := 0$ *a.e.* on Γ_{N}, $g_1 := 2$ *a.e.* on Γ_{N}, $g_2 \in \mathrm{L}^2(\Gamma_{\mathrm{N}})$ be the function defined by

$$\begin{array}{rccl} g_2 : & \Gamma_{\mathrm{N}} & \longrightarrow & \mathbb{R} \\ & (x,y) & \longmapsto & g_2(x,y) := x^2 - y^2, \end{array}$$

and $f \in \mathrm{L}^2(\Omega, \mathbb{R}^2)$ be the function defined by

$$\begin{array}{rccl} f : & \Omega & \longrightarrow & \mathbb{R}^2 \\ & (x,y) & \longmapsto & f(x,y) := \begin{pmatrix} \frac{5-x^2-y^2+xy}{4} & \frac{5-x^2-y^2+xy}{4} \end{pmatrix}. \end{array}$$

With $m := 2$, one has $g_1 \geq m$ *a.e.* on Γ_{N} and $0 < ||g_2||_{\mathrm{L}^\infty(\Gamma_{\mathrm{N}})} < m$, thus the assumptions from the beginning of Section 4 and from Subsection 4.3.1 are satisfied. We consider the initial control $z_0 \in \mathcal{U}$ given by

$$\begin{array}{rccl} z_0 : & \Gamma_{\mathrm{N}} & \longrightarrow & \mathbb{R} \\ & (x,y) & \longmapsto & z_0(x,y) := \cos(x^2 - y^2). \end{array}$$

We present now the numerical results obtained for the above two-dimensional example using the numerical methodology described in Subsection 4.3.1. Figure 2 depicts the control which solves Problem (4.1). It is a *bang–bang* optimal control, that takes exclusively the two values -1 and 1 on the boundary Γ_{N}. Figure 3 shows the evolution of the value of $\mathcal{J}$ with respect to the iteration. We observe an usual decreasing of the cost functional $\mathcal{J}$ with respect to the iteration.

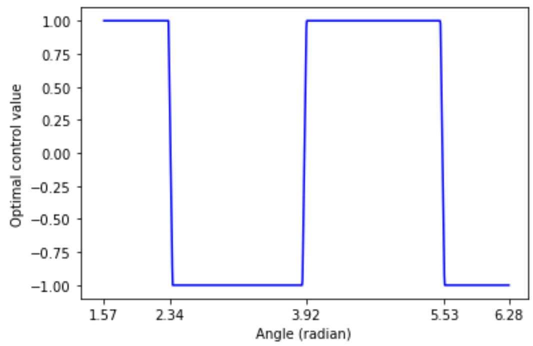

Figure 2: Values of the optimal control on the boundary Γ_{N}.

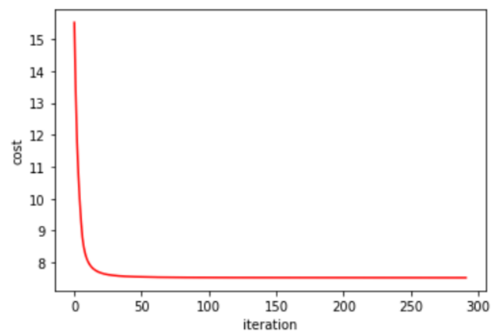

Figure 3: Values of the cost functional $\mathcal{J}$ with respect to the iterations.

References

[1] R. Adams and J. Fournier. *Sobolev Spaces*, volume 140 of *Pure and Applied Mathematics*. Elsevier, 2003.

[2] S. Adly and L. Bourdin. Sensitivity analysis of variational inequalities via twice epi-differentiability and proto-differentiability of the proximity operator. *SIAM Journal on Optimization*, 28(2):1699–1725, 2018.

[3] S. Adly, L. Bourdin, and F. Caubet. Sensitivity analysis of a Tresca-type problem leads to Signorini's conditions. *ESAIM Control Optim. Calc. Var.*, 28:Paper No. 29, 29, 2022.

[4] S. Adly, L. Bourdin, F. Caubet, and A. Jacob de Cordemoy. Shape optimization for variational inequalities: the scalar Tresca friction problem. To be published in SIAM Journal on Optimization (SIOPT), 2023.

[5] C. D. Aliprantis and O. Burkinshaw. *Principles of Real Analysis.* 3^{rd} Edition, Academic Press, New York and London, 1998.

[6] G. Allaire. *Analyse numerique et optimisation.* Mathématiques et Applications. Éditions de l'École Polytechnique., 2007.

[7] G. Allaire, F. Jouve, and A. Maury. Shape optimisation with the level set method for contact problems in linearised elasticity. *The SMAI journal of computational mathematics*, 3:249–292, 2017.

[8] H. Attouch. *Variational convergence for functions and operators*. Applicable Mathematics Series. Pitman (Advanced Publishing Program), Boston, MA, 1984.

[9] M. Biegert. On traces of Sobolev functions on the boundary of extension domains. *Proc. Amer. Math. Soc.*, 137(12):4169–4176, 2009.

[10] L. Bourdin, F. Caubet, and A. Jacob de Cordemoy. Sensitivity analysis of a scalar mechanical contact problem with perturbation of the Tresca's friction law. *J Optim Theory Appl*, 192:856–890, 2022.

[11] H. Brezis. *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, volume 5 of *North-Holland Mathematics Studies, No. 5. Notas de Matemática (50)*. North-Holland Publishing Co., Amsterdam, 1973.

[12] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, 2011.

[13] R. Dautray and J.-L. Lions. *Mathematical Analysis and Numerical Methods for Science and Technology: Volume 2: Functional and Variational Methods*. Springer-Verlag Berlin Heidelberg, 2000.

[14] C. N. Do. Generalized second-order derivatives of convex functions in reflexive banach spaces. *Transactions of the American Mathematical Society*, 334(1):281–301, 1992.

[15] G. Duvaut and J.-L. Lions. *Les inéquations en mécanique et en physique*. Travaux et Recherches Mathématiques, No. 21. Dunod, Paris, 1972.

[16] F. Hecht. New development in freefem++. *J. Numer. Math.*, 20(3-4):251–265, 2012.

[17] J.-B. Hiriart-Urruty and C. Lemaréchal. *Fundamentals of convex analysis*. Grundlehren Text Editions. Springer-Verlag, Berlin, 2001.

[18] A. Jacob de Cordemoy. Shape optimization for contact problem involving Signorini unilateral conditions. Submitted, 2023.

[19] N. Kikuchi and J. T. Oden. *Contact Problems in Elasticity*. Society for Industrial and Applied Mathematics, 1988.

[20] F. Kuss. *Méthodes duales pour les problèmes de contact avec frottement*. Thèse, Université de Provence - Aix-Marseille I, July 2008.

[21] A. Manzoni, A. Quarteroni, and S. Salsa. *Optimal control of partial differential equations—analysis, approximation, and applications*, volume 207 of *Applied Mathematical Sciences*. Springer, Cham, 2021.

[22] F. Mignot. Contrôle dans les inéquations variationelles elliptiques. *Journal of Functional Analysis*, 22(2):130–185, 1976.

[23] G. J. Minty. Monotone (nonlinear) operators in Hilbert space. *Duke Mathematical Journal*, 29(3):341–346, 1962.

[24] J. J. Moreau. Proximité et dualité dans un espace hilbertien. *Bulletin de la Société Mathématique de France*, 93:273–299, 1965.

[25] R. T. Rockafellar. On the maximal monotonicity of subdifferential mappings. *Pacific Journal of Mathematics*, 33(1):209 – 216, 1970.

[26] R. T. Rockafellar. Maximal monotone relations and the second derivatives of nonsmooth functions. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 2(3):167–184, 1985.

[27] R. T. Rockafellar and R. J.-B. Wets. *Variational Analysis*, volume 317 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag Berlin Heidelberg, 1998.

[28] J. Salençon. *Handbook of continuum mechanics (general concepts, thermoelasticity)*. Springer-Verlag, Berlin, 2001.

[29] A. Signorini. Sopra alcune questioni di statica dei sistemi continui. *Annali della Scuola Normale Superiore di Pisa - Classe di Scienze*, Ser. 2, 2(2):231–251, 1933.

[30] A. Signorini. Questioni di elasticità non linearizzata e semilinearizzata. *Rend. Mat. Appl., V. Ser.*, 18:95–139, 1959.

[31] J. Sokołowski. Sensitivity analysis of contact problems with prescribed friction. *Appl. Math. Optim.*, 18(2):99–117, 1988.

[32] J. Sokołowski and J. Zolésio. Shape sensitivity analysis of contact problem with prescribed friction. *Nonlinear Analysis: Theory, Methods & Applications*, 12(12):1399–1411, 1988.

[33] M. Tucsnak and G. Weiss. *Observation and Control for Operator Semigroups*. Birkhäuser Advanced Texts Basler Lehrbücher. Birkhäuser Basel, 2009.